A Puzzle about Sums

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Abstract

A famous mathematical theorem says that the sum of an infinite series of numbers can depend on the order in which those numbers occur. Suppose we interpret the numbers in such a series as representing instances of some physical quantity, such as the weights of a collection of items. A puzzle then arises: How do we interpret the metaphysical significance of this mathematical theorem? I argue that prior solutions to the puzzle lead to implausible consequences. Then I develop my own solution, where the basic idea is that the weight of a collection of items is equal to the limit of the weights of its finite subcollections contained within ever-expanding regions of space. I show how my solution is intuitively plausible and philosophically motivated, how it reveals an underexplored line of metaphysical inquiry about quantities and locations, and how it elucidates some classic puzzles concerning supertasks.

§1 A Puzzle

Let's start with a principle that initially seems unremarkable, yet that turns out to be puzzling:

> **SUM**
> For any collection of items, the weight of the collection equals the sum of the weights of the items within that collection.[1]

If the collection contains only finitely many items, then SUM is indeed rather boring. But if the collection is infinite, then a puzzle arises. The puzzle doesn't cast doubt on the truth of SUM. Instead, it raises questions about how to interpret the principle in the first place. As a prelude, consider the following thought-experiment (which I'll call '*Infinite Scale*') from Linnebo [2020: 189]:

> **Infinite Scale**
> Suppose you have a scale that is capable of weighing infinitely many items and an infinite amount of weight. Suppose also that you have an infinite number of iron balls and an infinite number of balloons. The first ball weighs 1 kg, the second ball weighs $\frac{1}{3}$ kg, the third ball weighs $\frac{1}{5}$ kg, and so forth. The first balloon lifts $\frac{1}{2}$ kg, the second balloon lifts $\frac{1}{4}$ kg, the third balloon lifts $\frac{1}{6}$ kg, and so forth. Now, suppose you first place the 1 kg ball on the scale, then attach the $-\frac{1}{2}$ kg balloon, then add the $\frac{1}{3}$ kg ball, then attach the $-\frac{1}{4}$ kg balloon, and so forth. This infinite sequence of actions results in an infinite progression of weights and counterweights added to the scale. What is the weight of the scale once every item has been added?[2]

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[1] Are collections sets? Well, sets are abstract objects, and abstract objects don't weigh anything. It may be better to think of collections as fusions or pluralities.

[2] Strictly speaking, weight is measured in *kgf* (kilograms-force), rather than *kg* (which is a unit of mass), so even a balloon wouldn't have negative weight (but instead would have buoyancy that counteracts the effect of weight on a scale). But since the paper that originally introduced this puzzle was framed in terms of weight, I'll set aside these points and continue to use weight as the target quantity in this paper. For a more carefully developed

To figure out the answer to the question, we need to sum the weights of all the individual items. The scale starts with nothing on it, so we start at 0. Then we add 1 (for the first ball), then subtract $\frac{1}{2}$ (for the first balloon), then add $\frac{1}{3}$, then subtract $\frac{1}{4}$, and so on. This infinite sequence of actions can be modeled by the alternating harmonic series:

The Alternating Harmonic Series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = ln(2) \approx 0.69.$$

The equation says that when we sum the sequence of numbers on the left, the result will gradually converge to $ln(2)$, which is approximately 0.69. Therefore, it seems reasonable to conclude that the weight on the scale at the end of the procedure is ~.69kg. But the puzzle arises when we ask the following question: what if we were to rearrange the items in the series?

A surprising result from mathematics is that merely rearranging the order of the terms in a series can result in convergence to a different sum. In other words, the sum of an infinite series sometimes depends on the order of its terms. In fact, an astonishing result is that by rearranging terms, we can make a series sum to any arbitrary real number, or even tend towards positive or negative infinity. This is known as "Riemann's Rearrangement Theorem," after its discovery by the 19th-century mathematician Bernhard Riemann.[3] Here's a statement of the theorem:

Riemann's Rearrangement Theorem

If a series is conditionally convergent, then its terms can be rearranged so that the new series converges to an arbitrary number, or diverges.

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scenario involving a physical manifestation of a conditionally convergent series (though in a classical mechanics setting) involving charge, see Alexander [forthcoming].

[3] Riemann [1876].

A series *converges* iff its sequence of partial sums approaches a finite limit.[4] If, for example, the series is $a + b + c + \ldots$, then its sequence of partial sums is $(a, a + b, a + b + c, \ldots)$. If that sequence approaches a finite limit x, meaning that the terms become arbitrarily close to x as the sequence progresses, then the series converges to x. Otherwise, the series *diverges*.

A series *conditionally converges* iff it converges but the series consisting of the absolute values of all its terms diverges. The alternating harmonic series described above is an example of a conditionally convergent series: it sums to $ln(2)$, but if we rearrange its terms, we can generate a different sum. To do so, we take terms from the original series until the sum reaches the number we want to converge on, and then alternate between positive and negative terms from the original series so that the rearranged series converges to the desired limit. For the purposes of this paper, it isn't necessary to go deeper into the mathematical reasoning behind Riemann's Rearrangement Theorem, though a brief explanation is in the APPENDIX.

The mathematical result is secure. The philosophical puzzle is how to interpret the metaphysical significance of the result. It's surprising that the sum of an infinite series depends on the order of its terms, but that may be regarded merely as a mathematical curiosity. What's much harder to believe is that the weight of a collection of items can depend on the order in which those items are weighed. Order of weighing, after all, seems a mere matter of convention. And there's nothing special about weight: as I'll discuss later, analogous puzzles arise with any quantity that satisfies certain formal conditions. So, we have a metaphysical puzzle. Let's call it the *puzzle of conditional convergence*.

The aims of this paper are to (**1**) explain why existing solutions to the puzzle are unsatisfactory, (**2**) develop a new solution, (**3**) support my solution by investigating an underexplored collection of metaphysical

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[4] More precisely, a series *converges* $=_{\text{def}}$ its sequence of partial sums approaches a limit. The *sequence of partial sums* is the sequence of the sums of each of the first n terms of the series, for every natural number n. If, for example, the series is $a + b + c + \ldots$, then its sequence of partial sums is $(a, a + b, a + b + c, \ldots)$. A sequence approaches a limit l just in case for every real number $\varepsilon > 0$, there there exists a natural number m such that for all $n \geq m$, the difference between l and the partial sum of the first n terms of the series is less than ε.

questions about quantities and locations, and (**4**) apply my results to some existing puzzles about supertasks. As we will see, the puzzle of conditional convergence isn't merely an isolated technical problem. Instead, the solution to the puzzle has much more general ramifications, yielding new lines of metaphysical inquiry and new tools for solving classic problems.

As for the puzzle itself, I'll argue that Riemann's Rearrangement Theorem indeed has interesting metaphysical implications. But it will take some work to uncover the exact nature of those implications. To solve the puzzle, we will need to consider the relationship between summation over individuals and summation over locations. Once we do so, the following solution will come to light: the weight of a collection of items (whether finite or infinite) is equal to the limit value of the weights of the finite subcollections contained within ever-expanding regions of space. The initial statement of this solution may feel a bit complex. But I'll argue over the course of the paper that the solution is intuitively plausible, philosophically motivated, and explanatorily fruitful.

The puzzle of conditional convergence doesn't require one to think of infinite scale scenarios as metaphysically possible. In fact, given the connections between weight, mass, and gravitational fields, I myself doubt that *Infinite Scale* is metaphysically possible. Instead, think of the thought-experiments as illustrative tools that allow us to more vividly assess some general questions concerning quantities, objects, and locations. As analogies, consider the roles that scenarios like Hilbert's Hotel, philosophical zombies, and Cartesian demons play in discussions of infinity, consciousness, and knowledge.[5]

The solution I develop will have implications for all quantities, even those that cannot generate puzzles of conditional convergence. And while the prime example in this paper will be weight (and space), the view I favor will generalize to quantities beyond weight (and locations beyond spatial regions). My goal is to not only develop a solution to the initial puzzle, but also to use the solution to motivate some bigger ideas about the

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[5] Even if infinite scales are metaphysically impossible, infinite sums of quantities may nevertheless be possible (or even actual). At least, it's an open possibility that the actual world contains infinitely many instances of some physical quantities.

metaphysical relationships between quantities and locations. Furthermore, I'll explain how pursuing this line of inquiry yields explanatory payoffs for some classic puzzles about supertasks, such as the Ross-Littlewood Paradox and Thomson's Lamp.

Here's the structure of the paper: **§2** argues against prior solutions to the puzzle, developed by Linnebo [2020] and Hoek [2023]; **§3** presents my view, which I call the 'expansionist analysis'; **§4** supports the expansionist analysis by exploring some general questions about quantities and locations; and **§5** applies the expansionist analysis to some puzzles about supertasks.

§2 The Order-Relative and Balance Analyses

Two solutions to the puzzle of conditional convergence already exist. The first, developed by Øystein Linnebo, is what I'll call the *order-relative analysis*: the weight of a collection depends on the order in which the individual items are weighed. The second solution, developed by Daniel Hoek, is what I'll call the *balance analysis*: the weight of a collection is zero whenever it contains both infinite positive weight and infinite negative weight. I'll argue that neither solution is satisfactory.

The Order-Relative Analysis

Since the puzzle was introduced by Linnebo [2020], it's fitting to start with his solution. Linnebo's view, in effect, is that we ought to take Riemann's Rearrangement Theorem at face value. According to his order-relative analysis, the weight of a collection of items depends on the order in which the individual items are weighed. The order-relative analysis predicts that in the scenario described above, the weight is approximately .69kg. But the order-relative analysis also holds that if the items were placed on the scale in a different order, then the result would be different.

The order-relative analysis is the most straightforward interpretation of the mathematical results. But the solution feels unsatisfying: it's very hard to believe that the order in which individual items are put on a scale can make a difference to the weight of the collection. In fact, we might wonder how weight works when our method of weighing doesn't involve any physical intervention on the items that are weighed.

Suppose, for example, that we already know the weights of the individual items, the items are already lying on the ground, and we calculate the weight of the collection by inputting the weights of the items into a calculator. It's implausible that the weight of the collection depends on the order in which we enter numbers into the calculator. And if you and I were to enter the weights of the items in different orders, would that mean that the collection would then have multiple weight values? The reason these consequences feel absurd is because weight isn't a matter of mere bookkeeping; it's an objective physical quantity. The solution I develop will preserve the order-invariance of weight (and other quantities).

Consider now a variant on *Infinite Scale*. The variant invokes an application of Riemann's Rearrangement Theorem. Recall that the weights of the items in *Infinite Scale* were mathematically represented by the alternating harmonic series. Since that series is conditionally convergent, there's a rearrangement that diverges to ∞.

Here's a procedure for achieving that result. First, we separate the positive terms (which I'll label the a_i's) from the negative terms (which I'll label the b_i's). Then we construct the new series as follows: we start with the first positive term a_1 (which is 1), followed by the first negative term b_1 (which is –1/2), followed by the subsequent positive terms $a_2, \ldots, a_n$ until the partial sum exceeds 2, followed by the second negative term b_2, followed by the subsequent positive terms $a_{n+1}, \ldots, a_m$ until the partial sum exceeds 3, and so forth. The result is a rearrangement of the alternating harmonic series with the following structure:

The Divergent Rearrangement

$$a_1 + (b_1 + a_2 + \ldots + a_n) + (b_2 + a_{n+1} + \ldots a_m) + \ldots \approx 1 + 1 + 1 + \ldots$$

This series diverges to ∞. We can then appeal to this divergent rearrangement to construct a variant on *Infinite Scale*:

Clustered Items

Items: The same collection of items as in *Infinite Scale*.

Setup: The temporal order in which the items are placed corresponds to the alternating harmonic series (so the same as in *Infinite Scale*). But the spatial arrangement of the items corresponds to the divergent rearrangement (so an iron ball, followed by a balloon, followed by many iron balls, followed by a second balloon, followed by many, many iron balls, and so forth).

Clustered Items and *Infinite Scale* are indistinguishable with respect to both (1) the items that comprise the collection, and (2) the order in which the items are placed on the scale. They differ only in (3) the spatial arrangement of those items. In *Clustered Items*, the iron balls are clustered together, and the balloons become sparse at an exponential rate; in *Infinite Scale*, every iron ball is adjacent to two balloons, and every balloon is adjacent to two iron balls. The order-relative analysis predicts that the weights of the collections are the same: namely, $ln(2)$. But while that result is plausible for *Infinite Scale*, it's not at all obvious for *Clustered Items*.

In fact, a natural answer for *Clustered Items* is that the weight is ∞.[6] What exactly this means is delicate. In standard number systems, ∞ isn't a value, but instead a symbol that's used when expressing limit behavior (such as when the output of a function grows without bound). But when thinking about infinitary scenarios, it's appealing to assign infinitary values, especially when the value of a quantity in a scenario seems to exceed all finite values. If we postulate infinitary values, however, then we face interesting and substantive questions about which transfinite number system to deploy and how the number system maps to the target quantities.

For the purposes of this paper, I'll stay neutral on how exactly to interpret the assignment of ∞. While this leaves open some big questions, the arguments I develop won't turn how we answer those questions. And while I'll eventually endorse the claim that the weight in *Clustered Items* is ∞, my

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[6] See Easwaran *et al* [2021: §3] for discussion of such issues.

present point is modest—I'm simply claiming it's unobvious that *Clustered Items* and *Infinite Scale* ought to be assigned the same weight value.

There's another clever argument against the order-relative analysis that comes from Hoek [2023]. Suppose we start with all the items already placed on the infinite scale, and then remove them—as follows—until the scale is empty:

Emptied Scale

Items: The same collection of items as in *Infinite Scale*.

Setup: We start at the end-state of *Infinite Scale*, where all items have already been placed and where (per the order-relative analysis) the resulting weight is .69kg. Then we remove the items as follows: first the 1kg ball *and* the $\frac{1}{3}$kg ball, then the $-\frac{1}{2}$kg balloon, then the $\frac{1}{5}$kg *and* the $\frac{1}{7}$kg ball, then the $-\frac{1}{4}$kg balloon, and so forth until all items have been removed from the scale.

As Hoek notes, the natural generalization of the order-relative analysis will entail that the weight after all the items have been removed is a negative number. But that's absurd, since at the end of the procedure there's nothing on the scale. This is a compelling reason against the order-relative analysis.

The Balance Analysis

According to Hoek [2023], the weight of the collection in *Infinite Scale* is 0. He appeals to the following principle:

> BALANCE
>
> If equal weights and counterweights lie on a scale, then the scale is in the same state as when it holds no weights.

Call this view the *balance analysis*. The basic idea is that for any collection of items, we can partition it into two equivalence classes, one containing the positively weighted items and the other containing the negatively weighted items. If the sum of the positive weights is equal to the inverse of the sum of the negative weights, then the weight of the whole collection is 0. To motivate the balance analysis, Hoek reinterprets the infinite scale (where the counterweights are balloons) as an infinite balance (where the weights are on the left side and the counterweights are on the right side). Here's the relevant passage, from Hoek [2023: 2]:

> **Infinite Balance**
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> Imagine an infinite stock of brass weights of 1kg, $\frac{1}{3}$kg, $\frac{1}{5}$kg, and so on; and an infinite stock of counterweights of $\frac{1}{2}$kg, $\frac{1}{4}$kg, $\frac{1}{6}$kg, and so on. At 1pm, we begin alternatively placing a weight on the left of our indestructible balance, and a counterweight on the right. We start with the biggest weights and work our way down...We speed up the steps as we go, so that at 2pm exactly, all steps have been performed. Which way will the balance lean after 2pm?

Here's the idea. Since both the left side and the right side of the balance contain an infinite amount of weight, it seems plausible that the balance will be in equilibrium. But *Infinite Balance* seems to be merely a redescription of *Infinite Scale*. Therefore, if *Infinite Balance* is in equilibrium, then we ought to think that *Infinite Scale* outputs 0kg. Here's that argument in premise-conclusion form:

The Balance Argument

P1: *Infinite Balance* is in equilibrium.

P2: If *Infinite Balance* is in equilibrium, then the weight in *Infinite Scale* is 0.

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C: The weight in *Infinite Scale* is 0.

Hoek [2023] focuses mainly on defending P1. This is because the order-relative analysis denies P1. Or, more precisely, it's in the spirit of the order-relative analysis to accept that it's possible for *Infinite Balance* to lean in one direction or the other (rather than to be in equilibrium), depending on the order in which the individual items are weighed. To defend P1, Hoek argues against the following principle, which he thinks of as the underlying motivation behind the order-relative analysis:

> **CONTINUITY**
> If a quantity converges to a limit x over time interval $[t_0, t_1)$, then the quantity attains value x at t_1.[7]

I won't argue against P1—I'll grant that *Infinite Balance* is in equilibrium. In fact, I agree with Hoek that CONTINUITY doesn't hold in all cases, and in §5 I'll provide a diagnosis of when CONTINUITY works and when it doesn't. Instead, I want to contest P2. On my view, *Infinite Balance* isn't merely an innocuous reinterpretation of *Infinite Scale*. The scenarios differ in ways that matter for how we assess the results for each case. As an initial challenge to the balance analysis, consider the following principle:[8]

> **FINITE ADDITIVITY**
> If a and b both have finite weight values, then the weight of a and b equals the weight of a plus the weight of b.

The balance analysis must deny FINITE ADDITIVITY. Let a be the collection of items in *Infinite Scale*, which the balance analysis says weighs 0, and let b be an additional iron ball that weighs 1. Given FINITE ADDITIVITY, the weight of a and b should be 1. But the balance analysis instead predicts that the weight of a and b is 0. In fact, this holds no matter how much the additional item weighs, and no matter how many additional items we add. I don't take this consideration to be decisive; unexpected results often occur when dealing with the infinite. But I think the violation of FINITE ADDITIVITY is at least a strike against the balance analysis, especially since the mathematical analogue of FINITE ADDITIVITY holds even for conditionally convergent series.

Here's another challenge to the balance analysis:

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[7] Principles of continuity—and in particular, extensions from finite to infinitary cases—are often attributed to Leibniz. See Jorgensen [2009].

[8] Assume a and b don't overlap—I'll briefly discuss overlapping objects later, in §4.

Heavy Items

Items:	An infinite number of elephants, each of which weighs 5000kg. An infinite number of balloons, each of which lifts 0.01kg.
Setup:	An elephant is placed on the scale, then a balloon is attached to that elephant, then a second elephant is placed on the scale, then a second balloon is attached to that second elephant, and so on.

Suppose we apply the balance analysis to *Heavy Items*. First, we partition the weights (the elephants) from the counterweights (the balloons). Then, we place all the elephants on one side of an infinite balance. Since balloons have negative weights, we need to find a kind of item for the other side of the balance whose positive weights exactly balance the negative weights of balloons.

Well, it's common knowledge that a standard helium balloon lifts approximately the weight of a slice of cheese. So, imagine that we replace each balloon with a slice of cheese, and then put all that cheese on the other side of the balance. Although one side contains elephants and the other side contains cheese, one might still conclude that the balance will be in equilibrium (since both sides contain an infinite amount of weight). But it's implausible that the weight in *Heavy Items* is 0: instead, it's much more plausible that the weight is ∞. Therefore, even if we assume that *Infinite Balance* is in equilibrium, we ought not thereby infer that *Infinite Scale* outputs 0.

A proponent of the balance analysis might contend that our finite imaginative capacities are leading us astray. Just because any finite number of elephants and balloons has a positive weight doesn't mean that an infinite number of elephants and balloons likewise has a positive weight. As an analogy, consider the intuition that there are fewer prime numbers than integers (when, in fact, both sets have the same cardinality). However, this error-theoretic explanation is unlikely to be an apt diagnosis of the present case. Any finite number of elephants would outweigh the same number of slices of cheese—yet I granted above that an infinite number of elephants may very well weigh the same as an infinite number of slices of cheese. This is evidence that the intuition behind *Heavy Items* is sensitive to the aforementioned asymmetries between finitary versus infinitary cases.

Besides, the argument can be strengthened. Here's a variant on *Heavy Items* that yields an especially forceful argument against the balance analysis:

Hungry Items

Items:	An infinite number of elephants, each of which weighs 5000kg. An infinite number of balloons, each of which lifts 0.01kg.
Setup:	An elephant is placed on the scale, then a balloon is fed to that elephant, then a second elephant is placed on the scale, then a second balloon is fed to that second elephant, and so on. Fortunately, the elephants have large gullets and flexible stomachs, so each balloon is swallowed whole by its elephant and remains inflated once inside its elephant's stomach.

The only difference between *Heavy Items* and *Hungry Items* is the relative locations of the elephants and balloons. In *Heavy Items*, the balloons are floating above the elephants; in *Hungry Items*, the balloons are inside the elephants. Unless we have independent reason for thinking that this change is relevant to the weights of the collections, we ought to treat *Heavy Items* and *Hungry Items* with parity. But surely the weight in *Hungry Items* is ∞. So, we ought to think that the weight in *Heavy Items* is likewise ∞. This indicates that we ought to reject BALANCE (and, consequently, the balance analysis).

We can now appreciate a more general problem for both the order-relative and balance analyses. Both solutions tacitly assume that we already know how to individuate the relevant items. However, when dealing with scenarios such as *Hungry Items*, it's unobvious how to do that. Should we count the balloons as separate from the elephants, or should each elephant (with a balloon inside) count as a single item? How we answer such questions will generate predictive differences for both the order-relative and balance analyses. Yet there seem to be no non-arbitrary answers. And we cannot simply permit any method of individuation whatsoever, since doing so would lead to contradiction.

We have now evaluated two solutions to the puzzle. Both initially appeared promising, but both turned out to be vulnerable to compelling counterarguments. Let's now turn to the solution I favor.

§3 The Expansionist Analysis

Here's the basic idea behind my view. To solve the puzzle of conditional convergence, we need to know not only the weights of the individual items, but also their spatial arrangement. More precisely, we need to know whether the finite subcollections of items contained within ever-expanding regions of space always converge to the same weight value. If so, then that's the weight of the collection. Otherwise, the collection's weight is either infinite or undefined. I'll call this the *expansionist analysis*.

The name 'expansionist analysis' is inspired by the view called 'expansionism' in the infinite ethics literature.[9] But while the views are similar in spirit, it's important to appreciate that the target issues are distinct (even when we set aside the superficial difference concerning whether value or weight is the target quantity). The core differences are that expansionist theories in infinite ethics aim to yield *comparisons* between worlds with *infinite* total values, whereas my concern is with *non-comparative* evaluations of infinitary collections of items that sum to *finite* total values. Because of this, the main question of this paper is logically orthogonal to the main question examined in the infinite ethics literature. Nevertheless, my view is a natural complement to expansionist theories in infinite ethics, and some of my later arguments (especially in §4) may indirectly support those theories.

The present section will focus mainly on explaining how the expansionist analysis works. But the full story—including the explanation for why weight and space are connected—will be developed also in §4, where I explore some general metaphysical questions about quantities and locations. Some of the discussion in this section will be a bit technical—for those

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[9] See Vallentyne & Kagan [1997], Bostrom [2011], and Wilkinson [2020].

less interested in that kind of stuff, it's possible to skim the technical details while still grasping the core ideas.

Definitions

Let a *ball* be the set of spatial points that lie within a given distance from some center.[10] To denote balls, I'll use the notation '$\mathrm{B}(p, d)$', where p is the ball's center and d is the ball's radius. For any collection A and ball $\mathrm{B}(p, d)$, we can identify the subcollection of A that lies inside $\mathrm{B}(p, d)$.[11] To denote this subcollection, I'll use the notation '$\mathrm{A}|\mathrm{B}(p, d)$'. If no items of A lie inside $\mathrm{B}(p, d)$, then $\mathrm{A}|\mathrm{B}(p, d) = \varnothing$; if every item of A lies inside $\mathrm{B}(p, d)$, then $\mathrm{A}|\mathrm{B}(p, d) = \mathrm{A}$.

Let ω be a function whose input is a collection of items and whose output is the weight of that collection. Hence, $\omega(\mathrm{A}|\mathrm{B}(p, d))$ is the weight of the subcollection of A that lies within the spatial ball with center p and radius d. For simplicity, I'll assume that any finite region of space contains only a finite number of weighted items (in §5, I'll discuss some cases where infinitely many items are in a finite region). Since we know how to calculate the weights of finite collections, and since any ball $\mathrm{B}(p, d)$ is finite, there will always be a straightforward answer as to the value of $\omega(\mathrm{A}|\mathrm{B}(p, d))$.

Procedure

The expansionist analysis says that the weight of collection A is x iff the weights of A's subcollections contained inside ever-expanding balls always converge to x, no matter which spatial point those balls are centered on. More precisely, define $\omega(\mathrm{A}) = x$ iff for all spatial points p, $\omega(\mathrm{A}|\mathrm{B}(p, d))$ approaches the limit x as d tends to ∞.

Here's a procedure for determining whether that condition is satisfied. We start by picking an arbitrary spatial point p and an arbitrary distance d. These two values determine a ball $\mathrm{B}(p, d)$—the ball with center p and

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[10] It doesn't matter whether the ball is open (excluding its boundary points) or closed (containing its boundary points).

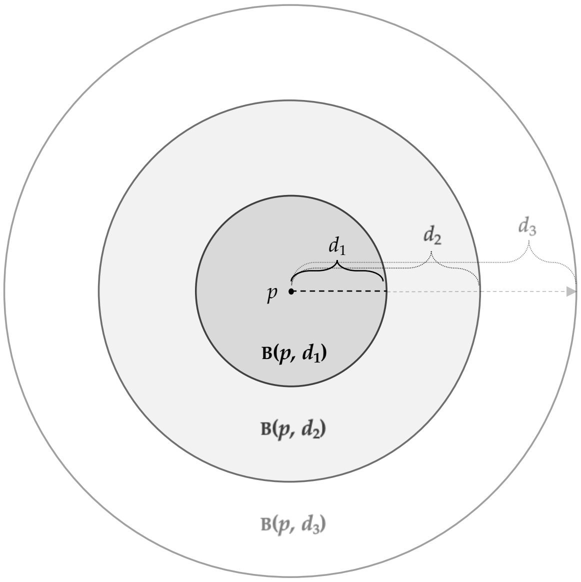
[11] What does it mean for an item a to be *inside* a region of space? Well, the answer doesn't matter. We can choose either (1) a is wholly inside the region, or (2) a is partially inside the region, or (3) a is mostly inside the region. Any of these choices will yield the same results (at least so long as the items are each finite in both weight and extent).

radius d. This in turn determines a set of items $\mathrm{A}|\mathrm{B}(p, d)$—the subcollection of A that lies within $\mathrm{B}(p, d)$. Then we ask: what happens to $\omega(\mathrm{A}|\mathrm{B}(p, d))$ as the ball grows larger?

To answer this, we define a sequence of balls that satisfies the following conditions: (1) every ball has the same center (namely, p), (2) every subsequent ball has a radius larger than the preceding ball, and (3) for every distance d_i, there's a ball in the sequence with radius d_n such that $d_n > d_i$. Put another way, the sequence of balls will be $(\mathrm{B}(p, d_1), \mathrm{B}(p, d_2), \mathrm{B}(p, d_3), \ldots)$. The first term denotes the ball with center p and radius d_1, the second term denotes the ball with center p and radius d_2, and so forth. Hence, we have a sequence of ever-expanding balls, each centered on the point p.

We then use this sequence of balls to define a corresponding sequence of weights: in particular, the sequence $(\omega(\mathrm{A}|\mathrm{B}(p, d_1)), \omega(\mathrm{A}|\mathrm{B}(p, d_2)), \omega(\mathrm{A}|\mathrm{B}(p, d_3)), \ldots)$. Here the first term denotes the weight of the subcollection of A that lies inside the ball with center p and radius d_1, the second term denotes the weight of the subcollection of A that lies inside the ball with center p and radius d_2, and so forth. Hence, we now have a sequence of the weights of the subcollections of A contained inside the sequence of ever-expanding balls (anchored on some center).

What we have defined so far is illustrated in the diagram below:



$$\begin{aligned}
\omega(\mathrm{A}|\mathrm{B}(p, d_1)) &= 1\text{kg} \\
\omega(\mathrm{A}|\mathrm{B}(p, d_2)) &= 1\text{kg} - \tfrac{1}{2}\text{kg} \\
\omega(\mathrm{A}|\mathrm{B}(p, d_3)) &= 1\text{kg} - \tfrac{1}{2}\text{kg} + \tfrac{1}{3}\text{kg} \\
\ldots &= \ldots
\end{aligned}$$

FIGURE 1: The weights of finite subcollections of A contained within ever-expanding spatial balls.

Let's call any sequence of weights derived through this kind of procedure an *ω-sequence*.[12] The ω-sequence illustrated by the diagram above is (1kg, $\frac{1}{2}$kg, $\frac{5}{6}$kg, …).

For any collection A, we can generate a set of ω-sequences. For simplicity, let's suppose the distance intervals are always fixed (this won't make a difference in the end). Then the ω-sequences for any given collection will be individuated by which spatial point p marks the center of the balls. In other words, for any collection A, there will be exactly one ω-sequence for every spatial point p. Now, for any ω-sequence, we can ask whether it approaches some limit x, meaning that the terms in the sequence get arbitrarily close to x as the sequence progresses. In other words, as the balls grow arbitrarily large, the weights of the subcollections of A within the balls become arbitrarily close to x.

Now I can state the core claim of the expansionist analysis. If every ω-sequence for A approaches the limit x, then the weight of A is x. If not, then the weight of A is either ±∞ or undefined. For the moment, I'll assume that every item in A is eventually captured by the ω-sequence (meaning that every item will eventually be contained within the ever-expanding sequence of balls). I'll later explain how to deal with cases where we drop that assumption.

The Expansionist Analysis

The expansionist analysis can be elegantly expressed with an equation (as reminders, A is a collection of items, x is a real number, and $\omega(\mathrm{A} \mid \mathrm{B}(p, d))$ is the weight of the subcollection of A that lies within the ball that has center p and radius d):

The Expansionist Analysis

$$\mathrm{A}\ \textit{weighs}\ x \quad \equiv \quad \forall p \lim_{d \to \infty} \omega(\mathrm{A} \mid \mathrm{B}(p, d)) = x$$

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[12] Note that 'ω-sequence' is usually used to denote sequences that have the ordering of the natural numbers. So, ω-sequences in the sense I've defined can be thought of as specific instances of ω-sequences in this more general sense.

The analysis says that the weight of a collection A is x iff for every spatial point, if we consider an ever-expanding sequence of balls centered on that point, then the weights of the finite subcollections of A contained with those balls will approach x. That's equivalent to saying all of A's ω-sequences approach x.

It's worth making a point about how the mathematics connects to the metaphysics. Recall that a series converges to x just in case its sequence of partial sums—the sequence of the sums of its first n terms, for every natural number n—approaches the limit x. Therefore, whether a series converges or diverges depends on the behavior of its associated sequence of partial sums.

Each of the solutions to the puzzle of conditional convergence offers a different answer as to *which* sequences of partial sums are relevant for determining the weight of a collection. The order-relative analysis says that the relevant sequence of partial sums corresponds to the order in which the items are weighed. The balance analysis identifies two relevant sequences of partial sums, one comprised of all the positively weighted items and the other comprised of all the negatively weighted items. And the expansionist analysis takes the relevant sequences of partial sums to correspond to the weights of the finite subcollections contained inside ever-expanding balls. The fact that these different answers are available illustrates how the mathematics underdetermines the metaphysics. Finding a solution to the puzzle of conditional convergence is a matter of identifying the right metaphysical interpretation of the mathematics.

I've now explained how the expansionist analysis works for cases involving conditional convergence. But what about cases involving divergence? If some of A's ω-sequences approach limit x while others approach limit y, then the weight of A is undefined. If each of A's ω-sequences tends to ∞ (or $-\infty$), then the weight of A is ∞ (or $-\infty$). Furthermore, while the expansionist analysis is motivated by the puzzle of conditional convergence (which involves infinitary collections), it generalizes to finite cases as well. If A is finite, then every item in A will eventually be contained within any sequence of ever-expanding balls, so every ω-sequence will eventually converge to exactly the weight of A.

Some might wonder why the expansionist analysis appeals to *uniform* expansions from a center. In principle, we could consider a more permissive restriction on expansions, such as a convexity or connectedness restriction. But it's plausible that uniformity is the relevant restriction, at least in the analysis of weight. If we were to instead adopt a more permissive constraint on expansions, then we would be faced with implausible predictions. Consider, for example, *Hungry Items*. Since there are gruesome sequences of spatial expansions where the associated partial sums in this scenario don't diverge to ∞, adopting a more permissive restriction on expansions would predict that the weight in *Hungry Items* is undefined rather than ∞. This strikes me as a good reason for rejecting a more permissive restriction on expansions for weight.

Are there other quantities for which non-uniform expansions are permissible? I suspect that when dealing with natural quantities, we will need to always restrict the analysis to natural expansions. But that's a speculation: to figure out the answer, we would need to undertake a more systematic exploration of the relevant expansionist analyses for other quantities. If there *are* other quantities for which it's appropriate to appeal to more permissive restrictions on expansions, though, then it should be easy enough to modify the expansionist analysis accordingly.

Isolated Items

The analysis above assumes that all the items in the collection are located within the same space. But what happens when we have isolated spaces?[13]

Let's say two items are *isolated* from each other if their distance is undefined. Imagine, for example, that a multiverse hypothesis is true,

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[13] Technically, it doesn't even make sense to sum weights across spatiotemporally isolated universes since there's no single gravitational field between isolated items. But analogous questions arise even when we consider other quantities (such as utility) where such questions do make sense. For the purposes of this subsection, I'll frame the discussion more neutrally and simply talk about the "values" of the items under consideration (without specification of any particular quantity). Afterwards, I'll return to weight.

where there are infinitely many spatiotemporally isolated universes (all of which are actual). And consider a finite collection of isolated items:

Finite Isolation

Items: Two iron balls, *a* and *b,* where *a* has quantity value 1 and is located in universe A, and *b* has quantity value 2 and is in universe B.

What's the total value of the collection comprised of *a* and *b*? Well, it's clear the answer ought to be 3. But no ever-expanding sequence of balls starting in *a*'s universe will ever reach any point in *b*'s universe, since the universes are spatially isolated. In fact, every ω-sequence starting in A will converge to 1, and every ω-sequence starting in B will converge to 2.

Fortunately, it's easy to deal with this sort of case. To determine the value of a collection that contains some isolated items, we partition the collection into a set of equivalence classes, where each equivalence class consists of all and only the items that aren't isolated from each other. Put another way, we group together items that inhabit the same space. Then we apply the expansionist analysis to each equivalence class, yielding a set of values. After that, we simply add up the values associated with each equivalence class to get a total value. In *Finite Isolation,* we apply the expansionist analysis twice—once to the subcollection in universe A, yielding a subtotal of 1, and again to the subcollection in universe B, yielding a subtotal of 2—and add up those subtotals, yielding a total of 3.

What justifies the procedure above? The answer to that question involves ideas that will be developed in the next section. But here's a preview. There are two distinct ways of summing quantities: (1) over a collection of individuals, and (2) over a collection of locations. In some cases, such as *Infinite Scale,* summation over individuals doesn't yield a definite verdict, since different orderings of the terms yield different sums. Yet we can still appeal to summation over locations (using the expansionist analysis) to find an answer. Conversely, in other cases, such as *Finite Isolation,* summation over locations doesn't yield a definite verdict, since some of the items are isolated from each other. Yet we can still appeal to summation over individuals (using FINITE ADDITIVITY) to find an answer.

In fact, even when we have an *infinite* collection of isolated items, we can still appeal to summation over the values of the individual items to find an answer, so long as the values either generate an absolutely convergent series or diverge to ±∞. But what if the values of a collection of isolated items generate a conditionally convergent series?

Infinite Isolation

Items: The exact same collection of items as in *Infinite Scale*, with every item located in a spatiotemporally isolated universe.

My view is that the value in *Infinite Isolation* is undefined. At least, I see no non-arbitrary way of summing the values of the items in this scenario. This might elicit the worry that 'undefined' is a non-answer. But I suspect that nearly everyone will want to accept that there are undefined values in at least some scenarios. Setting aside isolation cases, we can construct scenarios involving unbounded oscillating divergence, such as the series $1 - 2 + 3 - 4 + \ldots$. Those who reject 'undefined' as a possible answer must figure out what to say about these sorts of scenarios. And if we accept 'undefined' as the right answer in those cases, it's reasonable to think that it likewise applies in *Infinite Isolation*.

Verdicts about Cases

Let's return to the big picture. I'll now review how the expansionist analysis does better than the order-relative and balance analyses in generating plausible predictions. Here are the scenarios that have occurred throughout the paper (in the order in which they appeared), alongside the verdicts of the expansionist analysis:

Scenario	**Verdict**
Infinite Scale	*ln*(2)
Clustered Items	∞
Emptied Scale	0
Infinite Balance[14]	Equilibrium
Heavy Items	∞
Hungry Items	∞
Finite Isolation	3
Infinite Isolation	Undefined

I've already discussed the isolation cases. I'll briefly walk through the other predictions, starting with *Infinite Scale*.

Strictly speaking, *Infinite Scale* is under-described, since the scenario didn't specify the spatial arrangement of the items. But suppose the spatial arrangement of items corresponds to the temporal order in which those items are placed. Imagine, for example, that the first item is placed all the way on the left, the second item to the right of the first, the third item to the right of the second, and so forth. Then the expansionist analysis plausibly predicts (alongside the order-relative analysis) that the weight of *Infinite Scale* is ~.69. Next, consider *Infinite Balance*. It's uncontroversial that the weight on each side of the balance is ∞. If we then grant that a balance is in equilibrium just in case both sides carry the same amount of weight,[15] then we reach the result that *Infinite Balance* is in equilibrium.

The counterexamples to the order-relative analysis were *Clustered Items*, in which we arranged the items from *Infinite Scale* so that the iron balls were clustered together, and *Emptied Scale*, in which we removed all the items from *Infinite Scale* in a different order than they were placed. The order-relative analysis predicted that the weight of *Clustered Items* was ~.69

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[14] Here I interpret *Infinite Balance* as involving two collections of weights, one on either side of the balance.

[15] Actually, this principle is quite contestable. Consider: in the infinite ethics literature, many deny the analogous principle concerning value (so even if worlds A and B each have infinite value, it may nevertheless be that A is better than B). Perhaps analogous considerations apply to other quantities as well.

(instead of ∞) and that the weight on *Emptied Scale* would be negative (instead of 0). By contrast, the expansionist analysis plausibly predicts that the weight in *Clustered Items* is ∞ (since sequences of ever-expanding balls will generate sequences of ever greater weight) and that the weight in *Emptied Scale* is 0 (since there are no items on the scale when the scale is empty). Moreover, whereas the order-relative analysis takes weight to depend on what seems to be a mere matter of convention, the expansionist analysis (as we will soon see) appeals to independently motivated connections between quantities and locations.

The counterexamples to the balance analysis were *Heavy Items*, where infinitely many elephants had balloons attached to their backs, and *Hungry Items*, where infinitely many elephants had balloons inside their stomachs. The predictions of the balance analysis depend on how we individuate the items in the scenario. But on the most natural method of individuation, the balance analysis predicts that the weight in *Heavy Items* is 0 while the weight in *Hungry Items* is ∞. By contrast, the expansionist analysis treats these scenarios with parity: in both cases, the weight is ∞ (since in both cases, all sequences of ever-expanding balls generate sequences of ever greater weight). Moreover, whereas the balance analysis had to give up FINITE ADDITIVITY, the expansionist analysis can retain that principle.

Although I've argued that neither the order-relative nor the balance analyses are correct, I think that both still get something fundamentally right. The order-relative analysis is fundamentally correct that Riemann's Rearrangement Theorem has surprising metaphysical implications: by simply rearranging the items within a collection, we can change the weight of that collection. However, the order-relative analysis misidentifies the relevant parameter of rearrangement: it's spatial distribution, rather than temporal order, that matters. The balance analysis is fundamentally correct in rejecting CONTINUITY: just because the weights in a sequence of finite subcollections approach the value x before time t doesn't mean that the resulting infinite collection at time t weighs x. However, the balance analysis is too quick to dismiss the relevance of convergence to limits: it's convergence over regions of space, rather than intervals of time, that matters for weight. The expansionist analysis incorporates these lessons.

You might still feel uneasy about taking the value of a collection of items with respect to a quantity to depend on the spatial arrangement of those items. But if you deny any such connections, then you must also accept some uncomfortable consequences. To elicit this, consider an example that involves utility (rather than weight).

Imagine two scenarios where Hilbert's Hotel is fully occupied. In Scenario 1, the *odd*-numbered rooms are occupied by residents with +1 utility and the *even*-numbered rooms are occupied by residents with –1 utility. In Scenario 2, the *composite*-numbered rooms are occupied by residents with +1 utility and the *prime*-numbered rooms are occupied by residents with –1 utility. Since the occurrence of prime numbers grows increasingly infrequent as we move along the natural number line,[16] the positive utility is much more densely distributed in the second scenario. Yet the two scenarios may be thought of as mere spatial rearrangements, since every item in one scenario can be mapped to a corresponding item in the other scenario. If you think that spatial rearrangements never matter for summing quantities, then you will be forced to say that these scenarios involve equal total utilities. That strikes me as a terrible cost to incur.

The expansionist analysis has some surprising consequences. But everyone must accept some surprising consequences about infinitary scenarios, and I believe this is a consequence that we can get used to, learn to live with, and perhaps even come to love. Furthermore—as I'll discuss next—there's a deeper diagnosis of the connections between quantities and locations that will help us to make sense of this consequence.

§4 Quantities and Locations

In the expansionist analysis, physical space plays the role of the *locative category*: to determine the weight of a collection, we consider ever-expanding regions of space (rather than time, spacetime, or something else). Yet nothing in the formalism necessitates an appeal to physical space: in principle,

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[16] This follows from the prime number theorem, proved by Hadamard and de la Vallée Poussin in 1896. Interestingly, this theorem was also based on work by Riemann (in particular, the Riemann zeta function). See Weisstein [2022].

we could have instead appealed to ever-expanding temporal intervals, ever-expanding light cones, or ever-expanding regions of some other kind. In fact, anything with metric structure—the kind of structure associated with distances between elements—could (at least for mathematical purposes) be used for the locative category. Therefore, you might wonder: What justifies the connection between weight and space?[17]

This section aims to answer that question. By doing so, I'll also introduce some new lines of inquiry about the metaphysics of quantities and locations. At some points, I'll offer more questions than answers. But I think that's indicative of how much of the metaphysical terrain is underexplored.

Category Mistakes

Consider an asymmetry: it makes sense to ask how much weight is in a given region of space, but it doesn't make sense to ask how much weight is in a given region of time. Contrast 1*a* with 1*b*–1*d*:[18]

(1*a*) How much weight is in this room?
\# (1*b*) How much weight is in this hour?
\# (1*c*) How much weight is in the red region of color space?
\# (1*d*) How much weight is in the interval (0, 1)?

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[17] The ensuing discussion will assume a classical picture of space and time. One reason is merely to simplify the discussion. But another reason is that many of our concepts—including WEIGHT—are arguably classical concepts. That is, regardless of the actual metaphysics of space and time, WEIGHT bears different conceptual relations to SPACE than it does to TIME. Now, this raises some interesting questions about conceptual engineering: how should we adjust our concept of weight when we move to a relativistic framework? The natural options are to (1) appeal to frame-variant regions of space, or (2) appeal to frame-invariant regions of spacetime. I won't attempt to settle which of these options is best.

[18] These sentences are all formulated as questions, but other syntactic constructions (such as declarative sentences) would work just as well.

Whereas 1*a* is a sensible question, 1*b*–1*d* are category mistakes:[19] weight can be instantiated at regions of physical space, but it can't be instantiated at regions of other sorts of spaces. This asymmetry with respect to locations is analogous to a more familiar asymmetry with respect to individuals. Only certain kinds of entities—namely, material objects—have weight values. Other kinds of entities—colors, abstract objects, feelings, etc.—aren't the sorts of things for which it makes sense to ascribe weight values. Consider the asymmetry between 2*a* and 2*b*–2*d*:

(2*a*) How much does Riemann weigh?
(2*b*) How much does redness weigh?
(2*c*) How much does the number 3 weigh?
(2*d*) How much does love weigh?

Just as weight can be instantiated only *by* material objects, weight can be instantiated only *at* spatial locations. Even though the formalism for the expansionist analysis leaves open which locative category has the relevant metric structure, the interpretation of the formalism makes sense only if we take the locative category to be space. For any quantity, we can ask both about its *category of individuals* (which kinds of things can instantiate the quantity?) and its *category of locations* (which kinds of locations are those at which the quantity can be instantiated?). The answer to the latter question tells us which kinds of locations are relevant to the expansionist analysis for that quantity.[20]

Not every quantity has space as its locative category. Contrast weight with pain. While it doesn't make sense to ask how much weight occurred over an interval of time, it *does* make sense to ask how much pain occurred over an interval of time. If one feels pain for a longer duration,

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[19] 1*c* and 1*d* are especially odd because they commit an extra category mistake: only concrete objects can have weights, but concrete objects don't occupy regions of color space (rather, colors do) or regions of the real line $\mathbb{R}$ (rather, numbers do).

[20] This point is especially relevant to expansionist theories in the infinite ethics literature. A common criticism of these theories is that it's not clear why spacetime is relevant for value. My arguments here point to a response: spacetime is the locative category for value.

then more pain is instantiated. This indicates that time is a locative category for pain, even though it isn't for weight. Or contrast weight and number-of-prime-integers. While it doesn't make sense to ask how many primes there are in a given region of physical space, it *does* make sense to ask how many primes there are in a given interval of the number line. This indicates that while space is a locative category for weight, it isn't for number-of-primes.

Now, in one sense, weight values *are* indexed to times. If we ask how much weight is in a given region R (or instantiated by a given collection A), then we must specify the time at which we're evaluating the weight of R (or A). Otherwise, there won't be a determinate answer to the question, since the weight of a given region or collection may vary across different times. However, the way in which weight is indexed to time is different from the way in which weight is indexed to space (and to material objects). As we saw above, it doesn't make sense to sum weight over intervals of time, whereas it does make sense to sum weight over regions of space or collections of individuals. The specification of a time fixes the context of evaluation, rather than the domain of summation. Weight values are specified at particular times, but they aren't summed over temporal intervals.

It may turn out that some quantities are *locationless*, meaning they lack a locative category. If a quantity Q is locationless, then there are no answers to questions of the form 'Where is Q instantiated?' and no true sentences of the form 'Q is instantiated at R'. As a potential example, consider wealth. It's not obvious that it makes sense to ask how much wealth is instantiated within a given region of space or time (or any other locative category).[21] Similarly, it might turn out that some quantities are *objectless*, meaning they lack a category of individuals. As a potential example, consider number-of-prime-integers. While we can ask how many prime integers there are within a given region of the real line, it's not obvious that number of prime integers is instantiable by any individual entity (unless we interpret regions of the real line as themselves individuals).

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[21] If your linguistic intuitions differ, make sure that you aren't interpreting such expressions as elliptically asking how much wealth is instantiated by the collection of people within a given region of space or at a particular time.

If there are locationless quantities, then what happens when they generate puzzles of conditional convergence? Well, the expansionist analysis identifies restrictions on which sequences of partial sums are relevant for the sum of an infinite series. In particular, the relevant partial sums are the values of the finite subcollections inside ever-expanding balls of the relevant locative category. If a quantity is locationless, then there are no such restrictions. Therefore, it's natural to think that for these quantities, *all* sequences of partial sums are relevant. By consequence, when we construct a conditional convergence scenario for a locationless quantity, the value of the collection will be undefined.

What exactly *is* a location, anyway? This is a hard question—I'm optimistic about the prospects for a metaphysical analysis, but I don't have a settled answer. But I don't think we need one for present purposes. The connections between quantities and locations that I've identified are compatible with a range of views about the nature of locations. As examples, my arguments are intended to leave open whether objects and locations are mutually exclusive categories, whether locations are fundamentally absolute or relational, and how to best develop a formal theory of locations.[22]

The methodology I've applied to weight can be generalized. If R is a region that belongs to the category of locations for Q, then we should be able to sensibly ask 'How much of quantity Q is instantiated within region R?' If A is a collection of individuals that all belong to the category of individuals for Q, then we should be able to sensibly ask 'What is the value of collection A with respect to quantity Q?' The answers to such questions provide evidence as to the target quantity's category of locations and category of individuals.

Metaphysical Principles

I'll now turn to some metaphysical principles connecting quantities, locations, and individuals. The first principle will mitigate one of the dialectical burdens of the expansionist analysis, the second principle illustrates how

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[22] For some work on the nature of locations, see Casati & Varzi [1999], Hawthorne & Sider [2002], Parsons [2007], and Kleinschmidt (*ed*) [2014].

summation over locations behaves in systematic ways, and the third principle is a generalization of the initial summation principle introduced at the very beginning of this paper.

Suppose A is a collection of items and R is a region of space that contains all (and only) those items. Recall that ω is a function that takes as input a collection of items and outputs the weight of that collection. Let's generalize ω so that it can also take as input a region of space (where the output would then be the weight contained within that region of space). Here's a plausible principle about how the weights of collections of individuals relate to the weights contained within locations:

> **QUANTIFICATION EQUALITY**
> If A is the collection of weighted items in region R, then $\omega(A) = \omega(R)$.[23]

Our focus throughout the paper has been on questions about the weights of infinitary collections of items. But we could have instead focused on questions about the weights contained within infinitary regions of space. Let A be one of the infinitary collections of items we have considered (such as the collection in *Infinite Scale*) and R be the region of space that contains all and only those items. QUANTIFICATION EQUALITY entails that the answers to the following questions should be the same:

Q1: What is the weight of A?
Q2: What is the weight within R?

The equivalence of these questions matters for the puzzle of conditional convergence. According to the expansionist analysis, we can answer Q1 by finding the limit of the weights of the finite subcollections of A contained within ever-expanding regions of space. It's natural to then ask why regions of space are relevant for calculating the weight of a collection of individuals. But that question feels less compelling when we shift the focus from Q1 to

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[23] According to *supersubtantivalism*, material objects are identical to regions of spacetime. If supersubstantivalism is true, then QUANTIFICATION EQUALITY trivially follows. But even if supersubstantivalism is false, QUANTIFICATION EQUALITY is plausible.

Q2. If we ask how much weight is contained within some region R, it seems obvious that the answer can be found by summing the weights contained within the subregions of R. But QUANTIFICATION EQUALITY entails that Q1 and Q2 will have the same answer. Hence, if it's permissible to appeal to space to answer Q2, it should also be permissible to appeal to space to answer Q1.

QUANTIFICATION EQUALITY also enables the expansionist analysis to avoid a problem that beset the order-relative and balance analyses. Recall that both analyses were subject to a problem about how to individuate the items within the collection. The problem was illustrated via *Hungry Items*, when we asked whether the elephants and the balloons counted as separate items or whether each elephant and the balloon inside of it counted as a single item. On the expansionist analysis, however, it doesn't matter how we answer such questions. This is because of QUANTIFICATION EQUALITY. For any region of space, there's some determinate answer as to the amount of weight contained within that region, no matter how we individuate the collection of items contained within that region.

The second metaphysical principle I want to consider concerns sums over the weights contained within regions. Suppose we already know the weights contained within two regions of space, R1 and R2. What, then, is the weight contained within the union of R1 and R2? The following answer is plausible:

SUMS OVER REGIONS

$\omega(R_1 \cup R_2) = \omega(R_1) + \omega(R_2) - \omega(R_1 \cap R_2)$.

This principle yields the right verdicts across different cases. There are three possibilities for how R1 and R2 may be related: (1) R1 and R2 are identical, (2) R1 and R2 are disjoint, and (3) R1 and R2 overlap (where the overlap is partial, and where this includes cases where one region wholly contains the other). Here's what the principle says for each case:

(1) **Identity:** The weight in $R_1 \cup R_2$ is the weight contained within either region. That is: $\omega(R_1 \cup R_2) = \omega(R_1) = \omega(R_2)$.

(2) **Disjointness:** The weight in $R_1 \cup R_2$ is the weight of R_1 plus the weight of R_2. That is: $R_1 \cap R_2 = \varnothing$, so $\omega(R_1 \cap R_2) = 0$, so $\omega(R_1 \cup R_2) = \omega(R_1) + \omega(R_2)$.

(3) **Overlap:** The weight in $R_1 \cup R_2$ is the weight of the disjoint part of R_1, plus the weight of the disjoint part of R_2, plus the weight of the intersection of R_1 and R_2. That is: $\omega(R_1 \cup R_2) = \omega(R_1 \setminus R_2) + \omega(R_2 \setminus R_1) + \omega(R_1 \cap R_2)$.

It's worth comparing SUMS OVER REGIONS to the corresponding principle concerning sums over individuals. We've already encountered a version of the latter principle: it was introduced at the beginning of the paper, under the simple label 'SUM', and it stated that for any collection of items, the weight of the collection equals the sum of the weights of the items within that collection. That principle is plausible if we assume that none of the items in the collection overlap with each other. But we can also generalize that principle so that it applies even when the items overlap.

Consider, for example, a statue and the clay that constitutes it, which are distinct but overlapping objects. If we're calculating the weights of material objects, then (in most contexts)[24] we wouldn't want to double-count the weight of the statue and the weight of the clay. Put another way, the weights of overlapping material objects are quantitatively redundant. To capture this precisely, we can construct a principle for sums over individuals that's structurally analogous to the principle for sums over regions. Whereas our second metaphysical principle ranged over two regions R_1 and R_2, our third metaphysical principle ranges over two individuals *a* and *b*:

SUMS OVER INDIVIDUALS

$$\omega(a, b) = \omega(a) + \omega(b) - \omega(a \cap b).$$

An interesting question is how the intersection relation in SUMS OVER INDIVIDUALS relates to the intersection relation in SUMS OVER REGIONS. One

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[24] There may be some unusual contexts where we would want to count the weights of the statue and the clay separately: perhaps, for example, in certain metaphysics seminars. But in these contexts, we would be deploying a non-standard method for summing weights.

option is to hold that they're the same: for two individuals to intersect just is for them to intersect in their spatial locations. Another option is to hold that they differ: mereological intersection is distinct from spatial intersection. I won't take a stance on this issue. But it's worth noting that those who favor the second option face an explanatory challenge. Given QUANTIFICATION EQUALITY, SUMS OVER REGIONS, and SUMS OVER INDIVIDUALS, two individuals overlap just in case their spatial regions overlap. Those who postulate two overlap relations must explain why these relations systematically coincide.

Generalizing the Puzzle

To generate a puzzle of conditional convergence, a quantity must be (1) *summative*, meaning that the quantity value of a collection is the sum of the quantity values of the individuals within that collection, (2) *convergeable*, meaning that the quantity values can be arbitrarily close to zero,[25] and (3) *polar*, meaning that the quantity has both positive and negative values. These properties are formally specified below (let ω be a function from an individual to its quantity value, ε be a real number, and each v_i be a quantity value):

SUMMATIVE: $\omega(a, b) = \omega(a) + \omega(b) - \omega(a \cap b)$
CONVERGEABLE: $\forall \varepsilon > 0, \exists v_n(|v_n| < \varepsilon)$
POLAR: $\forall v_1 \exists v_2(v_1 + v_2 = 0)$

None of these conditions expresses a necessary property of all quantities. As examples, it's arguable that (*a*) volume is summative and convergeable

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[25] Convergeability is tricky. In most (maybe all) cases, only a finite set of values can be instantiated by the kinds of objects that actually exist. But that's compatible with thinking that there are possible values of the quantity that aren't instantiated in the actual world. For example, even if there's a minimal weight instantiated by actual objects, there may still be smaller possible weights that aren't instantiated by any actual objects.

but not polar, (*b*) wealth is summative and polar but not convergeable,[26] (*c*) height-above-sea-level is polar and convergeable but not summative, and (*d*) temperature isn't summative, convergeable, or polar.

Philosophical work on quantities has focused mostly on what quantities have in common (and what distinguishes them from non-quantitative properties). But the differences above illustrate the diversity that exists within the domain of quantities. An interesting project would be to identify the most important features that differentiate various kinds of quantities and to generate a natural taxonomy.

§5 Supertasks

Many philosophical puzzles concerning infinitary scenarios involve *supertasks*—scenarios where an infinite number of steps are completed within a finite amount of time. The goal of this last section is to illustrate how the expansionist analysis sheds light on puzzles about supertasks. To start, let's return to a principle that was mentioned earlier:

> **CONTINUITY**
> If a quantity converges to a limit x over time interval $[t_0, t_1)$, then the quantity attains value x at t_1.

Hoek [2023] warns that CONTINUITY isn't a reliable guide to the outcomes of supertasks. He says: "We cannot uncritically apply the Continuity Principle…[T]he answer is different in each case…[E]ach supertask raises its own, subject-specific set of questions" (*p*.4). I think Hoek is right that CONTINUITY doesn't always yield the right results. But I also think that we can systematically diagnose when the principle holds and when it doesn't. The answer depends on whether time is a locative category for the quantity under consideration.

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[26] This assumes that wealth is measured in a currency with a minimal value: for example, the minimal unit for US dollars is 1 cent. If we instead consider an infinitely divisible currency, such as bitcoin, then wealth may be convergeable.

Any supertask will take place over some interval of time [t_0, t_1), such that the supertask begins at t_0 and is complete at t_1. To apply the expansionist analysis to a supertask, we need to consider increasingly large temporal intervals (t_j, t_k) such that (1) for all j, t_j is identical to or after t_0, and (2) for all k, t_k is before t_1. In other words, we appeal to ever-expanding intervals of time that approach (but don't reach) the start and end times of the supertask. However, before we get to that point, we must first ask whether the expansionist analysis is even appropriate for the supertask at hand. To do that, we need to figure out whether the supertask involves summation over some quantity, and if so, ask whether time is a locative category for that quantity. In what follows, I'll discuss three supertasks and show how each warrants relevantly different analyses.[27]

Infinite Flea

Items: A flea jumping around on a continuous line.

Setup: Each position on the line corresponds to a real number. The flea starts at position 0. At 1pm, it jumps 1cm to the right. At 1:30pm, it jumps $\frac{1}{2}$ cm to the left. At 1:45pm, $\frac{1}{3}$ cm to the right. And so forth.

Just as with *Infinite Scale*, the movements of the flea can be modeled by the alternating harmonic series: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = ln(2)$. Linnebo [2020] suggests that *Infinite Flea* is merely another case where the result of an infinitary scenario depends on the order in which the individual items are

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[27] Another philosophical puzzle is the Pasadena game, introduced by Nover & Hajek [2004]. Imagine you're presented with the following game: a coin is flipped n times, where n = the first flip where the coin comes up heads. If n is odd, then you receive $2^n/2$ dollars. If n is even, then you pay $2^n/2$ dollars. How much should you be willing to pay to play this game? If we calculate the expected value for each value of n, then we encounter a familiar series: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = ln(2)$. But Nover & Hajek argue that any order by which we sum these expected values is arbitrary, leading them to conclude that the value of the Pasadena game is indeterminate. The Pasadena game raises some tricky issues that warrant more discussion than the other puzzles discussed in this section. For limits of space, I won't address it in this paper.

evaluated—instead of iron balls and balloons, the relevant items are now left-jumps and right-jumps. This means that the order-relative analysis endorses CONTINUITY (at least for *Infinite Flea*). Since we've seen that appeals to temporal ordering relations can yield implausible results in some infinitary scenarios, it's reasonable to be suspicious of this way of reasoning about *Infinite Flea*. But this suspicion can be assuaged by observing an important asymmetry between *Infinite Scale* and *Infinite Flea*.

Think of the quantity in *Infinite Flea* as distance traveled, where jumps to the right are positive distances and jumps to the left are negative distances. To elicit the asymmetry between *Infinite Scale* and *Infinite Flea*, consider the contrast between the following questions:

(3*a*) How much distance was traveled in this interval of time?
\# (3*b*) How much weight is this interval of time?

Although time isn't a locative category for weight, it *is* for distance traveled. In other words, it makes sense to sum distance traveled (but not weight) over intervals of time. Given this, we ought to expect CONTINUITY to yield the right results for *Infinite Flea*. This is because if we take time to be the relevant locative category, then an invocation of the expansionist analysis is effectively an invocation of CONTINUITY. Both CONTINUITY and the expansionist analysis appeal to convergence to a limit: it's just that CONTINUITY requires that the limit be defined over intervals of time, whereas the expansionist analysis leaves open which locative category is relevant. For *Infinite Flea*, the question becomes whether the amount of distance traveled via the finite subsets of jumps occurring over ever-expanding intervals of time always approaches some limit x. The answer is 'yes': that limit is $ln(2)$.

Consequently, the order-relative analysis happens to be right about *Infinite Flea:* the total distance traveled by the flea depends on the order in which the flea makes its jumps. But the expansionist analysis gives a deeper explanation for why the order-relative analysis works in this case: namely, because time is the relevant locative category in this scenario. However, that condition may not hold for other supertasks involving other quantities. In fact, we will now turn to one such case.

The Ross-Littlewood Paradox

Items: A jar that can hold infinitely many balls, and a countably infinite pile of balls, numbered 1, 2, 3, and so forth.

Setup: At t_0, we place balls 1–10 into the jar. Then we remove ball 1. Then we add balls 11–20 into the jar. Then we remove ball 2. We repeat indefinitely. By time t_1, every ball from the original infinite pile has been placed in the jar.

How many balls are in the jar once the supertask is complete? If we appeal to CONTINUITY, then it seems that we should conclude that the answer is ∞. For every time before t_1, the number of balls in the jar grows increasingly large. If we mathematically represent the number of balls that are added to or removed from the jar at each step, we get the series $10 - 1 + 10 - 1 + \cdots$, which clearly diverges to ∞. However, Littlewood [1953] and Ross [1976] both argue that the answer is 0. The reason is that every ball is eventually removed from the jar. That is, for every ball in the jar, there will be some time before t_1 when that ball is removed. If one were to say that the number of balls at t_1 is ∞, we could ask *which* balls are in the jar at t_1. But any ball we pick would eventually be out of the jar at some time, so it seems that there's no ball such that it remains in the jar at t_1.

The expansionist analysis offers insight into the Ross-Littlewood paradox. The quantity under consideration is number-of-balls. Should we expect the value of this quantity at t_1 to be the limit value of the quantity for the times before t_1? Well, we can ask whether time is a locative category for number of balls:

\# (4*a*) How many balls are in this interval of time?
(4*b*) How many balls are in this region of space?

The asymmetry is evidence that time *isn't* a locative category for number-of-balls. Therefore, we ought to refrain from appealing to CONTINUITY when assessing the Ross-Littlewood Paradox. Now, that's merely a negative result: it doesn't yet settle how many balls are in the jar at t_1. But it undermines the main motivation for thinking that the answer is ∞. On the other hand, the observation that every ball is eventually removed from the jar remains untouched. Given this, I think the most reasonable answer to the Ross-Littlewood Paradox is 0.

Let's turn now to the last supertask:

Thomson's Lamp

Items: A lamp.

Setup: The lamp is turned on at 1:00pm, then off at 1:30, on at 1:45, and so on.

What's the state of the lamp at 2pm? Most philosophers who have thought about Thomson's Lamp argue that the scenario is under-described. I agree; I think the description of the scenario simply leaves open whether the lamp is on or off at 2pm. But set that aside: I want to instead make a more general point about how we reason about scenarios like Thomson's Lamp.[28]

It's often thought that Thomson's Lamp can be modeled by the infinite series $1 - 1 + 1 - 1 + \cdots$, commonly known as 'Grandi's Series'. But Thomson's Lamp differs in an important respect from the other scenarios we've considered: the relevant variable in Thomson's Lamp—whether the lamp is on or off—isn't a quantity. Consider how the puzzle would work just as well if the lamp flips back and forth between red light and green light, or if a screen alternates between displaying 'A' and 'B'. In such cases, there's no obvious analogue of addition or subtraction. Because of this, we should be cautious about using Grandi's Series to model Thomson's Lamp. If we wish to represent a scenario with an infinite series, then we should first ensure that the scenario involves modulations of some quantity. Otherwise, the algebraic operations may lack any meaningful interpretation,

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[28] This scenario originates from Thomson [1954], which is also where the term 'supertask' was first introduced. The idea that the scenario is underspecified is often associated with Benacerraf [1962].

and we risk conflating features of the mathematical representation with features of the scenario being represented.[29]

Granted, there *is* an obvious quantity that we could focus on in this scenario: namely, luminosity. We might interpret the luminosity level as 1 when the lamp is on and 0 when the lamp is off. Then the question becomes whether time is a locative category for luminosity. Does it make sense to ask how much luminosity there is over an interval of time?

I myself feel unsure in this particular case. But let's conditionalize. If time *isn't* a locative category for luminosity, then that means that even when we reinterpret Thomson's Lamp as involving modulations in luminosity levels, it's *still* inappropriate to use Grandi's Series to model Thomson's Lamp. On the other hand, if it does make sense to sum luminosity over time, then it's clear that Grandi's Series will be the appropriate mathematical representation for this case. And since Grandi's Series diverges, we might then conclude that the luminosity level of Thomson's Lamp at 2pm is undefined.

Now, it's actually possible to resist that conclusion. Throughout this paper, I've taken for granted the standard definition of the sum of an infinite series, where the sum is the limit of the sequence of partial sums of that series. But there are more powerful mathematical methods that assign finite numbers even to series that diverge under standard summation.[30] For Grandi's Series, nearly every one of the more sophisticated summation methods assign a sum of ½. And that brings us to a whole new philosophical question about infinite sums: which mathematical method for summation best captures the metaphysics of physical quantities?

I think that's a fascinating question. But it's a question that will have to be reserved for another time.

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[29] As a cautionary example, consider this passage from Thomson [1954: 6]: "[T]he reading-lamp has either of two light-values, 0 ('off') and 1 ('on'). To switch the lamp on is then to add 1 to its value and to switch if off is to subtract 1 from its value. Then the question whether the lamp is on or off after the infinite number of switchings have been performed is a question about the value of the lamp after an infinite number of alternating additions and subtractions of 1 to and from its value, i.e. is the question: What is the sum of the infinite divergent sequence +1, -1, +1, ...?"

[30] See Hardy [1992].

Conclusion

The puzzle of conditional convergence may initially seem like a remote curiosity. But the solution to the puzzle is surprisingly consequential. Though I've focused on weight, the puzzle generalizes to any quantity that's summative, convergeable, and polar. And the solution I favor—the expansionist analysis—has metaphysical implications well beyond our initial puzzle. I've explained how it allows us to identify when CONTINUITY yields the right verdicts in a supertask (and when it doesn't). And while this paper has been framed as a puzzle in metaphysics, the ideas I've developed are also applicable to puzzles in infinite ethics and infinite decision theory.

To properly solve the puzzle of conditional convergence, we needed to explore the general connections between quantities and locations. Only then were we in position to fully appreciate the metaphysical significance of Riemann's Rearrangement Theorem. On the picture I've developed, quantities are indexed to both categories of individuals (namely, the individuals that can bear values along that quantity) and categories of locations (namely, the locations at which that quantity can be instantiated). And summation over locations and summation over individuals interact in systematic ways, as illustrated by QUANTIFICATION EQUALITY, SUMS OVER REGIONS, and SUMS OVER INDIVIDUALS.

To my knowledge, there has been little prior philosophical investigation into the relationships between quantities and locations. The philosophical literature on quantities tends to focus on the structural features that distinguish quantities from other kinds of properties and on the ontology of quantities. The philosophical literature on locations tends to focus on the formal principles connecting locations to mereology and on the debates between substantivalists and relationalists. Hence, the puzzle of conditional convergence—and the expansionist analysis, in particular—points towards a line of metaphysical inquiry that is ripe for exploration.

There are many interesting questions that remain open. I've focused on weight, but we might also ask what the relevant locative categories are for other quantities. I've focused on metric spaces, but we might also ask whether the expansionist analysis can be extended to more general spaces, such as topological spaces. I've focused on uniform expansions, but we might also ask whether there are quantities for which non-uniform expansions yield the right results. I've focused on infinitary regions, but we might also generalize the expansionist analysis to cases where infinitely many items lie within a finite region. And I've assumed that summation is simply a matter of the limits of partial sums, but we might ask whether a more powerful mathematical method for summation is more metaphysically apt. These all strike me as promising lines for future research.[†]

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APPENDIX: Riemann's Rearrangement Theorem

A *sequence* is an ordered list of numbers. A *series* is the operation of summing the terms in a sequence. The *sequence of partial sums* of a series is the sequence where the n^{th} term (for every natural number n) is the sum of the first n terms of the series.

A series *converges* iff there exists a real number l such that the sequence of partial sums of the series converges to l. Equivalently: for any real number $\varepsilon > 0$, there exists a natural number m such that for all $n \geq m$, the absolute difference between l and the partial sum of the first n terms of the series is less than ε. A series *diverges* iff it doesn't converge.

Some series are *absolutely convergent*, meaning that the order of the terms in the series doesn't make any difference to the sum. Other series are *conditionally convergent*, meaning that the sum of the series depends on the order of its terms. More precisely, a series is conditionally convergent $=_{\text{def}}$ the series converges yet its absolute series (consisting of the absolute values of all of its terms) diverges. That is:

Definition: $\sum a_n$ *conditionally converges* $=_{\text{def}}$ $(\exists l : \sum a_n = l)$ and $(\neg\exists l : \sum |a_n| = l)$.

As an example, consider again the alternating harmonic series:

The Alternating Harmonic Series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = ln(2)$$

This series converges to $ln(2)$. But it's conditionally convergent: if we take the absolute values of its terms, then the resultant series $|\,1\,| + |-\frac{1}{2}\,| + |\,\frac{1}{3}\,| + |-\frac{1}{4}\,| + \cdots$ diverges to ∞. Notably, whenever an infinite series is conditionally convergent, the series containing all and only its positive terms diverges to ∞, and the series containing all and only its negative terms diverges to $-\infty$. That fact follows from the definition of 'conditional convergence', and will be important in what follows. Next, let's turn to Riemann's Rearrangement Theorem:

> **Riemann's Rearrangement Theorem:**
> If an infinite series is conditionally convergent, then its terms can be rearranged so that the new series converges to an arbitrary number, or diverges.

In what follows, I'll explain how to make the rearranged series sum to an arbitrary positive number or diverge to ∞. It will be straightforward to generalize to the cases involving arbitrary negative numbers and −∞.

To start, let's extract from the alternating harmonic series the subseries consisting of all and only its positive terms and the series consisting of all and only its negative terms. Note that for both the positive subseries and the negative subseries, the terms get arbitrarily close to zero as each subseries progresses. That is, $\forall \varepsilon > 0, \exists n \forall m \in \mathbb{N} \ (m \geq n \rightarrow |a_m| < \varepsilon)$.

The Positive Subseries

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots = \infty$$

The Negative Subseries

$$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \cdots = -\infty$$

Convergent Rearrangements

Suppose we wish to rearrange the alternating harmonic series so that it converges to an arbitrary positive number l. We start by taking terms from the positive subseries until the sum tips above l. That is, letting each a_i be a term from the positive subseries, we select the smallest n such that $a_1 + a_2 + \ldots + a_n > l$. Then we take subsequent terms from the negative subseries until the sum dips below l. That is, letting each b_i be a term from the negative subseries, we select the smallest m such that $(a_1 + a_2 + \ldots + a_n) + b_1 + b_2 + \ldots + b_m < l$.

Then we continue the procedure, taking the subsequent terms from the positive subseries whenever the sum dips below l, then the subsequent terms from the negative subseries whenever the sum tips above l, and so on. Since the terms in both the positive subseries and the negative subseries get arbitrarily close to zero, the sequence of partial sums in the rearranged series gets arbitrarily close to *l*.

The result is a rearranged series that contains all and only the terms of the original series, yet which converges to *l*. The sequence of partial sums oscillates around *l* with shrinking amplitude (and hence converges to *l*).

Divergent Rearrangements

Suppose we wish to rearrange the alternating harmonic series so that it diverges to ∞. Start by taking the first term a_1 from the positive subseries. Since the first positive term is 1, the partial sum is (trivially) 1. Then add the first term b_1 from the negative subseries followed by the subsequent positive terms a_2, a_3, …, a_n until the sum $b_1 + a_2 + \ldots + a_n > 1$. Now the partial sum exceeds 2. Then add the second term b_2 from the negative subseries followed by the subsequent positive terms a_{n+1}, …, a_m until the sum $b_2 + a_{n+1} + \ldots + a_m > 1$. Now the partial sum exceeds 3. This procedure is repeated indefinitely.

Since the positive subseries diverges to ∞, it's guaranteed that our remaining set of positive terms will suffice to exceed any finite positive integer, no matter how far along we are in the procedure. The result is a rearranged series that contains all and only the terms in the original alternating harmonic series, yet which diverges to ∞.

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