

# Dimensions *as* Quotients

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## Abstract

A *dimension* is—roughly—a way of varying with respect to a kind. For example, hue is a dimension of color. This paper develops a systematic analysis of the dimension relation. The core idea is that for a property  $D$  to be a dimension of a property  $F$  is for the space of  $D$ -values to be a *quotient* of the space of  $F$ -values. A quotient is a universal mathematical operation that involves collapsing some distinctions while preserving some structure. I explain how this analysis yields intuitive verdicts and recovers a variety of structural principles connecting dimensions and their kinds. Then I argue that the dimension relation is the “metaphysical dual” to the parthood relation, and I develop an important but underexplored distinction between dimensions and “parameters.”

## Introduction

A *dimension* is—roughly—a way of varying with respect to some kind.<sup>1</sup> As examples, consider the following statements:<sup>2</sup>

- a. HUE is a dimension of COLOR.
- b. SPEED is a dimension of VELOCITY.
- c. LONGITUDE is a dimension of GEOLOCATION.
- d. TIME is a dimension of SPACETIME.
- e. EXTRAVERSION is a dimension of PERSONALITY.

The concept DIMENSION shows up in all sorts of contexts: ordinary discourse, scientific models, and nearly every area of philosophy. Within recent philosophical work, dimensions have played prominent roles in metaphysics, mind, ethics, aesthetics, language, and history.<sup>3</sup> If we were to rank structural concepts by how universal or flexible they are, then DIMENSION would rival canonical concepts such as LOCATION, MAGNITUDE, or PART.<sup>4</sup>

Yet surprisingly, there has been little philosophical work on dimensions. While the concept is frequently used, it's rarely analyzed. This leaves a lacuna in the literature. What, exactly, do we mean when we say that some D is a dimension of some F?

If our use of 'dimension' were undisciplined and unsystematic, then that question might not even be worth investigating. If a definition of 'dimension' could be offloaded to another field—such as mathematics or physics—then there may be no need for philosophical analysis. But I'll argue that the subject-matter of dimensions is ripe for philosophical exploration and promising in its payoffs. The goal of this paper is to develop an analysis of dimensions that's intuitive, natural, formalizable, and fruitful.

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<sup>1</sup> Other terms with nearby meanings include 'degrees of freedom', 'direction of movement', 'parameter', 'coordinate', 'variable', and 'aspect'.

<sup>2</sup> There are some important differences across these examples. But I've purposefully chosen a diverse sample to illustrate the target relation. In some of my other work—especially on dimensionalization—I explore some of the relevant differences.

<sup>3</sup> As examples, see Funkhouser [2006] (determinables), Green [2020] (the perception-cognition border), Kulvicki [2020] (pictorial semantics), Hedden & Munoz [2024] (value), Hedden & Nebel [2024] (social choice theory), Mason [2025] (fittingness of emotions), Strawson [2025] (physical reality), White [2025] (aesthetic judgment), Maley [2026] (analog representation), Melamedoff [2026] (Spinoza's metaphysics), Jacobson [forth.] (valence), Kirby [forth.] (aesthetics), and D'Ambrosio & Stoljar [ms] (consciousness).

<sup>4</sup> In §6, I argue that dimensions and parts stand in a "duality" relationship: each is a basic way of extracting something simple from something complex. This provides a more concrete justification for taking DIMENSION and PART to be of equal metaphysical significance.

My central idea is that for a property  $D$  to be a dimension of a property  $F$  is for the space of  $D$ -values to be a quotient of the space of  $F$ -values. A quotient—informally—is a way of simplifying a structure by collapsing distinctions. I’ll argue that this analysis satisfies a number of desiderata for a theory of dimensions, and I’ll identify a deep connection between the dimension relation and one of the most general concepts in mathematics.

Here’s the plan: §1 clarifies the target sense of ‘dimension’, §2 outlines some structural principles as desiderata for a theory of dimensions, §3 explains how partitions are a useful tool for thinking about dimensions, §4 develops the core theory of dimensions as quotients, §5 discusses formal properties of the dimension relation, §6 argues that there’s an elegant “metaphysical duality” between the dimension relation and the parthood relation, §7 distinguishes multidimensionality from other kinds of “multiplicity structures,” and §8 draws a philosophically significant structural distinction between dimensions and “parameters.” The APPENDIX explains—in mostly informal terms—the mathematical relationship between quotients and partitions.

## §1 The Target Concept

Let’s start with a basic distinction (I’ll henceforth use ‘*wrt*’ for ‘with respect to’):

a <i>dimension</i> of $F$	a way of varying <i>wrt</i> $F$
the <i>dimensionality</i> of $F$	the number of ways of varying <i>wrt</i> $F$

This distinction in terminology maps to a distinction in questions:

<b>ways-of-varying?</b>	What is it for some $D$ to be a dimension of some $F$ ?
<b><math>n</math>-dimensionality?</b>	What is it for some $F$ to be $n$ -dimensional?

This paper focuses principally on the ways-of-varying question. In other work, I examine the  $n$ -dimensionality question in detail. These questions, of course, interact with each other—and I’ll discuss some of those interactions later in this paper. But the questions can also be investigated separately, and my answer to the ways-of-varying question won’t depend on any particular answer to the  $n$ -dimensionality question.

When we talk about dimensions and dimensionality, we're typically talking about properties.<sup>5</sup> For example, hue (a property) is a dimension of color (a property), and color is three-dimensional (a property of a property). This makes the following line of reasoning tempting:

- (1) Dimensions are *properties*.  
 (2) Kinds are *properties*.  
 (3) Dimensionality is a *property of properties*.
- Hence, (c) Dimensions are *properties of properties*.

But this inference is incorrect. Consider:

**Observation:** Hue is a dimension of color. But hue isn't a property of color. Instead, it's a property of *colored things*. When we say that D is a dimension of F, we aren't committing ourselves to the claim that F itself has a D-value, but instead to the claim that the bearers of F-values are also bearers of D-values.<sup>6</sup>

For any property F, the **values** of F are the ways of being F. For two individuals to differ with respect to F is for them to differ in their F-values. I'll assume the values of F are its maximal determinates. For example, the values of color are maximally determinate colors (red<sub>34</sub>, green<sub>17</sub>, etc.).<sup>7</sup>

The values of any property generate a **space**, meaning a set (whose elements are the values) equipped with some structure (capturing relations such as similarity and magnitude). For example, the similarity relations between colors and the magnitude relations between masses determine the structures of color space and mass space. These

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<sup>5</sup> I'll assume that kinds are determinable properties. But my arguments work even if we assign kinds (or dimensions) a different ontological status. For example, we also sometimes ascribe dimensions to concepts (COLOR), adjectives ('colored'), nouns ('color'), and spaces (color space).

<sup>6</sup> If a property F is itself a property of properties, then any of its dimensions D will likewise be a property of properties. But this is the exception, rather than the rule.

<sup>7</sup> The values of F should be distinguished from the realizers of F. For example, every color value is multiply realizable, in the sense that there are multiple distinct microphysical states of affairs that can realize that color value. If a surface is red<sub>34</sub>, and we make a minor alteration to the atomic configuration of that surface, then the surface will (probably) still be red<sub>34</sub>. But this doesn't mean that red<sub>34</sub> itself has multiple values. Those distinct realizers don't make a difference to the way the surface is with respect to *color* (even if they make a difference to the way the surface is with respect to microphysical configuration). Typically, even the maximal determinates of a property will be multiply realizable.

spaces can have various kinds of mathematical structures, including topologies, metrics, algebraic operations, and orderings.<sup>8</sup> The figure below illustrates the geometry of the color disk (which captures hue and saturation but abstracts away from brightness):

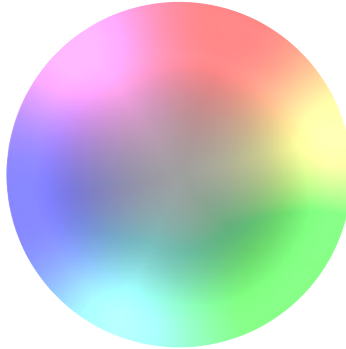


FIGURE 1: The color disk.

Throughout the paper, I'll sometimes move back and forth between mathematical and metaphysical language. For example, I'll talk about partitions over properties, distances between values, and one property being a quotient space of another. This is common in philosophical discussions of property spaces, and will streamline some of the prose.

### Senses of 'Dimension'

Since 'dimension' is used in a number of different ways, it's useful to explain how my target sense of 'dimension' (ways of varying *wrt* some kind) relates to some other uses of the term.

**Spaces:** Alongside properties, it's common to take the bearers of dimensions to be spaces. As examples, the real line is one-dimensional, the Cartesian plane is two-dimensional, Newtonian space is three-dimensional, and Minkowski spacetime is four-dimensional. While spaces and properties are distinct ontological categories, there's a natural translation scheme when thinking about dimensions. For any space—whether physical, mathematical, or otherwise—there's an associated property: namely, location within that space. The dimensions of a space  $s$  thereby correspond to the dimensions of

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<sup>8</sup> What explains the structure of a property space? On realist views, the structure is grounded in the nature of the property itself. On anti-realist views, the structure is grounded in our representations of the property. I'll speak like a realist throughout the paper, but nothing essential hangs on this. It's uncontroversial that color, mass, and other properties are associated with spaces of values, even if it's controversial how to understand the nature of those spaces.

the property location-within-s.<sup>9</sup> Conversely, whenever we talk of dimensions of a property, we can also talk about dimensions of the associated property space. For example, the property color is associated with the space of color values. Given this translation scheme, I'll freely move back and forth between talking about dimensions of  $F$  and dimensions of the space of  $F$ -values.

**Dimensional Analysis:** In physics, 'dimension' is most prominent in dimensional analysis, a method for analyzing the relationships between different physical quantities. In these contexts, the *dimension* of a quantity  $Q$  (*a*) specifies the basic quantities from which  $Q$  is derived, (*b*) abstracts away from specific units of measurement, and (*c*) is expressed as a product of powers. For example: 'The dimension of velocity is length over time (L/T).'<sup>10</sup> You might be tempted to think that the basic quantities in the dimensional analysis sense of 'dimension' just are the ways of varying with respect to the target quantity. But the example of velocity shows this isn't correct: neither length nor time are ways of varying with respect to velocity. Instead, the dimensions of velocity—in the sense I'm interested in—are speed and direction. I'll later argue that if  $D$  is a dimension of  $F$ , then anything with an  $F$ -value has a  $D$ -value. But notice that things with velocity values (material objects) aren't things that have length values (at least not in the relevant sense) or time values.

**Dimensions vs. Parameters:** In some philosophical contexts, the expression 'dimension of  $F$ ' has been used to mean (roughly) an input to the function that determines  $F$ -values. For example, if welfare is a function of pleasure, knowledge, and friendship, then pleasure, knowledge, and friendship have been called "dimensions of welfare." I'll later explain why this is distinct from my target sense of 'dimension', and I'll disambiguate by using the term 'parameter' to express this other notion. In §8, I'll discuss the relationship between dimensions and parameters in more detail, and I'll explain how the distinction clarifies a variety of philosophical questions.

**Mathematics:** In mathematics, the term 'dimension' has various definitions, depending on the kind of mathematical object under consideration.<sup>11</sup> But in each case, the

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<sup>9</sup> A related sense of 'dimension' concerns the *spatial measurements* of an object: 'The dimensions of the box are 2"×12"×7.' Less often, 'dimension' is used to refer to "parallel planes of existence": 'This alien came from the 5<sup>th</sup> dimension!'. I think both these senses relate to the sense of 'dimension' I'm interested in, but I won't try to analyze the connections.

<sup>10</sup> See Jalloh [2023] and Jacobs [2024] on dimensional analysis.

<sup>11</sup> The most prominent include definitions for *vector spaces* (number of basis vectors needed to generate the space), *topological spaces* (smallest  $n$  such that each point has a neighborhood homeomorphic to  $\mathbb{R}^n$ ), *graphs* (smallest dimensionality of Euclidean space in which the graph, with unit edges, is representable), *partial orders* (smallest number of total orders whose intersection generates the partial order), and *rings* (longest chain of prime ideals).

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definition specifies a measure over a class of mathematical objects, which outputs a number (specifying the dimensionality of that mathematical object). These definitions are obviously relevant to the  $n$ -dimensionality question.<sup>12</sup> But they have no obvious bearing on the ways-of-varying question. Even if we know how to measure the dimensionality of the space of  $F$ -values, we won't automatically know which properties are dimensions of  $F$ . To answer that question, we need a better understanding of the structural principles connecting dimensions and their kinds. That's the topic I'll turn to next.

## §2 Ways of Varying

Occasionally, I'm asked what it is for something to be a dimension simpliciter.<sup>13</sup> But I find myself unsure of what is being asked. For the sense of 'dimension' I'm targeting, questions about dimensions should be relativized to kinds. Instead of asking the monadic question of what it is for something to be a dimension, we ought to instead ask the relational question of what it is for some  $D$  to be a dimension of some  $F$ .

I'll use **dimension relation** to express the relation in virtue of which one property is a dimension of another. To embark on our exploration of the dimension relation, let's start with some basic observations that can serve as desiderata for an analysis. From the fact that hue is a dimension of color, we might infer the following:

- (1) Everything with a color value has a hue value.
- (2) Nothing with a single color value has multiple hue values.
- (3) All differences in hue are differences in color.
- (4) Some colored things differ in hue values.
- (5) Not all differences in color are differences in hue.

I believe these observations capture some of the structural connections between dimensions and kinds. These connections can be expressed schematically:

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<sup>12</sup> However, for reasons I discuss in other work (on the  $n$ -dimensionality question), even the answer to that question cannot simply be offloaded to mathematics.

<sup>13</sup> In some cases, the idea is that for  $D$  to be a dimension is for  $D$  to have a certain kind of intrinsic structure, such as magnitude or metric structure. But as I discuss later, there's arguably no kind of internal structure that's essential for  $D$  to be a dimension of  $F$ .

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UNIVERSALITY	Everything with an F-value has a D-value.
INCOMPOSSIBILITY	Nothing with a single F-value has multiple D-values.
VARIABILITY	All D-differences are F-differences.
? ASPECTUALITY	Not all F-differences are D-differences.
? MULTIPLICITY	Some things with F-values differ in D-values.

I've put question marks beside MULTIPLICITY and ASPECTUALITY because those principles concern edge cases. In particular, MULTIPLICITY rules out dimensions that have just a single value, and ASPECTUALITY rules out dimensions that are structurally equivalent to their kinds. While both principles are intuitive, I think the choicepoint of whether to accept them ought to turn partly on questions about how to best formalize the dimension relation. As a comparison, consider how similar choicepoints arose for whether to define 'part' and 'subset' reflexively or irreflexively. I'll return to this point in §5.

### Structural Principles

I'll now briefly motivate each principle mentioned above, continuing to use color as my main example.

**UNIVERSALITY:** Let *orange* be a determinate of color that corresponds to a proper subset of color space. Intuitively, orange isn't a dimension of color; instead, it's a determinate of color. If your initial intuitions diverge, then ensure that you aren't thinking instead of *orangeness* (meaning the degree to which an object is orange, which *is* intuitively a dimension of color) rather than orange (meaning a certain region of color space). Whereas every colored object has an *orangeness* value, not every colored object has an orange value. For example, something that's absolute black has an *orangeness* value (presumably zero), but it doesn't have an orange value (since absolute black isn't a value within the orange region of color space). Without UNIVERSALITY, we wouldn't be able to secure this result.<sup>14</sup>

**INCOMPOSSIBILITY:** This principle ensures that the values of a dimension (*a*) belong to the same category, and (*b*) don't overlap with each other. For example,

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<sup>14</sup> You might worry that UNIVERSALITY precludes some ordinary ascriptions of dimensions. It's natural, for example, to say that cardiovascular strength is a dimension of health. Yet plants—which lack hearts—can still be healthy. However, I think such cases are better explained by quantifier domain restriction: the initial context is implicitly restricted to humans, rather than all organisms. If we were to give up UNIVERSALITY, then we would be faced with an unpalatable proliferation of dimensions. For example, just as plants don't have hearts, humans don't have roots, tentacles, or thoraxes, and so don't have values with respect to root structure, tentacle length, or thorax density. But it seems unreasonable to count all those as dimensions of health (at least if we're considering all organisms with health values).

INCOMPOSSIBILITY precludes red, sweet, and loud from each counting as distinct values along a dimension (since an individual can be red, sweet, and loud). It also precludes red<sub>34</sub>, red, and colored from each counting as distinct values along a dimension (since red<sub>34</sub>, red, and colored overlap). In fact, INCOMPOSSIBILITY arguably follows from a more general principle concerning properties: namely, that the maximally determinate values of a property must exclude each other.

**VARIABILITY:** Let hue × shape be the set of conjunctions of hue values and shape values. For example, the values of hue × shape include red ∧ square, red ∧ round, green ∧ square, and so forth. Intuitively, hue × shape isn't a dimension of color. This is because shape is independent of color. Hence, not all variations in hue × shape are variations with respect to color. If  $x$  is red ∧ round and  $y$  is red ∧ square, then  $x$  and  $y$  differ with respect to hue × shape, but  $x$  and  $y$  don't differ with respect to color. To satisfy the idea that dimensions are ways of varying with respect to a property, we need VARIABILITY.<sup>15</sup>

**ASPECTUALITY and MULTIPLICITY:** As mentioned earlier, ASPECTUALITY and MULTIPLICITY rule out edge cases. ASPECTUALITY captures the idea that dimensions are always more *coarse-grained* than their kinds. MULTIPLICITY captures the idea that dimensions must be ways of *varying*. In every paradigm example where some  $D$  is a dimension of some  $F$ , both ASPECTUALITY and MULTIPLICITY are satisfied. Yet as I noted earlier, there may nevertheless be some theoretical advantages to relinquishing these principles. For the time being, however, I'll continue treating these principles as desiderata for a theory of dimensions.

### Intrinsic Structure

The principles above specified structural relations between dimensions and their kinds. But you might wonder whether there's any kind of intrinsic structure that's essential to dimensions. Suppose all we know is that  $D$  is a dimension of some  $F$ . What can we thereby infer about the structure of  $D$ ?

Many invocations of dimensions implicitly assume some answer to this question. For example, it's frequently assumed that dimensions must be degreed, or have

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<sup>15</sup> You might worry that VARIABILITY can't accommodate cases where changes along multiple dimensions cancel each other out. Consider, for example, an axiology where moral value is determined by welfare and equality, and where an increase in welfare can be exactly compensated by a decrease in equality, meaning that two worlds might be exactly the same in value yet have differing proportions of welfare and equality. This may appear to be a case where a  $D$ -difference (welfare) needn't yield an  $F$ -difference (value). But I'll later argue that this conflates dimensions with what I'll call 'parameters', where the parameters of  $F$  are the inputs to the function that determines  $F$ -values. The axiology described above is better classified as a view where value is unidimensional but has multiple parameters.

zero values, or have metric structure. But none of these kinds of structures is built into the concept DIMENSION. I'll illustrate using a few classes of counterexamples.

**Multidimensional Dimensions:** Some dimensions are themselves multidimensional. While this may initially sound slightly paradoxical, it aligns with our ordinary and theoretical uses of 'dimension'. For example, direction is a dimension of velocity, fitness is a dimension of health, and openness is a dimension of personality. Yet direction, fitness, and openness are each multidimensional. As a more philosophical example, consider two-dimensional semantics: it's obvious that both the primary and secondary dimensions of meaning are themselves multidimensional. Dimensions are ways of varying *wrt* a kind, and ways of varying can themselves be multidimensional.

**Non-Linear Dimensions:** Some dimensions have cyclical (rather than linear) structures. For example, hue—a dimension of color—is commonly taken to have the structure of a circle. And direction—a dimension of velocity—has the structure of a sphere. This means that there are no canonical orderings or privileged zero points amongst hue values and direction values. Therefore, dimensions needn't have degree structure or zero values.

**Ordinal Dimensions:** Some dimensions have ordinal (rather than interval, ratio, or absolute) structure.<sup>16</sup> For example, educational degree (BA, MA, PhD, etc.) might be taken to be a dimension of academic achievement. But there's no sense in asking whether the distance between PhD vs. MD is greater than the distance between MA vs. MSc. Educational degree has ordinal structure, rather than a stronger measurement scale.

**Nominal Dimensions:** Some dimensions have merely nominal structure, meaning the space of values is merely an unstructured set. For example, suppose we're deciding on which electronic files to transfer, and the relevant factors are file format (PDF, .docx, etc.) and file size (number of *kb*). Then it's natural to say that file format is one dimension of file type, even though there may be no additional structure on the space of file formats.

**Categorical Dimensions:** Some dimensions are categorical, meaning they have just two values (naturally thought of as "on"/"off"). For example, suppose we're US border agents deciding who to let into the country, and the only criteria guiding our decisions are (*a*) whether the person is a US citizen, and (*b*) the person's net worth. In such a context, it's natural to say that citizenship is a dimension of immigration qualifications, even though citizenship is a categorical property.

**The General Lesson:** If all we know is that *D* is a dimension of *F*, then we can infer almost nothing about the intrinsic structure of *D*-space. This is methodologically

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<sup>16</sup> See Stevens [1946] for the standard taxonomy on measurement scales.

important. It indicates that the interesting structural properties of dimensions are extrinsic—relating dimensions and their kinds—rather than intrinsic. This also means that a general analysis of dimensions ought not assume that the properties have any particular kind of intrinsic structure. Instead, we need to think about abstract structural properties that are applicable to any kind of mathematical space.

### §3 Partitions

Here’s an observation that will serve as a stepping stone for my eventual analysis:

**Observation:** Whenever  $D$  is a dimension of  $F$ ,  $D$  *partitions* the space of  $F$ -values. For example, hue partitions color: it’s possible to divide color space into cells, where each cell consists of all color values associated with the same hue value.

A **partition** is a way of dividing a set into a collection of disjoint non-empty subsets such that each element is in exactly one of those subsets. Partitions are generated by an **equivalence relation** (a relation that’s reflexive, transitive, and symmetric). Each equivalence class—called a “cell”—consists of exactly the elements that stand in that equivalence relation to each other. The diagram below depicts a partition on a disk-shaped space:

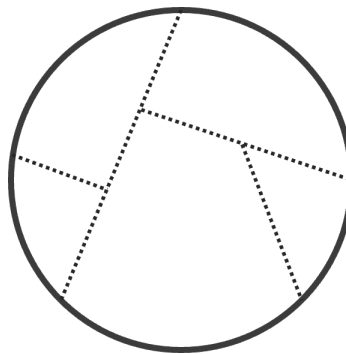


FIGURE 2: A partition.

As a mathematical example, the natural numbers  $\mathbb{N}$  can be partitioned by *parity*, meaning whether a number is odd or even. Two natural numbers  $n$  and  $m$  stand in the relevant equivalence relation just in case either both are even or both are odd. Hence, this equivalence relation places every natural number into either the even cell or the odd cell. To denote the equivalence relation that generates a partition, it’s standard to use

the symbol  $\sim$ . To denote the space that results from partitioning  $\mathbb{N}$  by  $\sim$ , we write  $\mathbb{N}/\sim$ . Hence,  $\mathbb{N}/\sim$  is the division of  $\mathbb{N}$  into two cells, “odd” and “even.”

A property  $D$  partitions a property  $F$  just in case  $D$  generates an equivalence relation over the space of  $F$ -values. The relation is equivalence with respect to  $D$ . That is,  $F$ -values  $F_1$  and  $F_2$  are equivalent *wrt*  $D$  just in case  $F_1$  and  $F_2$  determine the same  $D$ -value.<sup>17</sup> For example, one way of partitioning color space is by equating all color values that are exactly the same in hue (abstracting away from differences in saturation and brightness). Each cell in the partition then corresponds exactly to a hue value. A partition over  $F$ , in general, is a way of abstracting over certain differences in  $F$ -values.

There’s a close relationship between dimensions and partitions. Whenever  $D$  is a dimension of  $F$ ,  $D$  generates a partition over  $F$ -space. This invites the following view:

**The Partition Analysis**

$D$  is a dimension of  $F$  :=  $D$ -space is a partition of  $F$ -space.

The partition analysis is especially attractive when we see that it can satisfy *all* of the earlier criteria for a theory of the dimension relation. Consider the following relationships between the elements of a space (which correspond to  $F$ -values) and the cells of a partition (which correspond to  $D$ -values):

UNIVERSALITY	Each element is in exactly one cell.
INCOMPOSSIBILITY	No element is in multiple cells.
VARIABILITY	Different cells determine different elements.
MULTIPLICITY	Some elements are in distinct cells.
ASPECTUALITY	Not all elements are in distinct cells. <sup>18</sup>

Yet the partition analysis cannot be the end of the story. Partitions are defined in merely set-theoretic terms. But in nearly all cases of interest, both the kinds and the dimensions under consideration aren’t merely sets, but instead **spaces**, meaning sets

<sup>17</sup> I’ll understand the entailment relation intensionally: for  $F$ -values to entail  $D$ -values is for it to be the case that necessarily, any two things with the same  $F$ -value must have the same  $D$ -value. This is equivalent to the claim that  $D$ -values metaphysically supervene on  $F$ -values. Accordingly, two  $F$ -values  $F_1$  and  $F_2$  entail the same  $D$ -value just in case necessarily, anything with  $F_1$  has the same  $D$ -value as anything with  $F_2$ .

<sup>18</sup> To satisfy MULTIPLICITY and ASPECTUALITY, we need to exclude the trivial partitions. In particular, MULTIPLICITY requires excluding the trivially coarse partition that equates all  $F$ -values, while ASPECTUALITY requires excluding the trivially fine partition where each cell contains only a single  $F$ -value.

equipped with some structure. Moreover, whenever some  $D$  is a dimension of some  $F$ , the structure of  $D$ -space seems systematically related to the structure of  $F$ -space. A theory of dimensions ought to explain how to think about these structural connections. This indicates that the partition analysis is still missing a key piece of the puzzle.

Nevertheless, the partition analysis is fundamentally on the right track. There's a natural way of strengthening it that satisfies the above desideratum. The solution lies in the concept of a quotient.

## §4 Quotients

A quotient—informally—is a way of collapsing together elements to generate a simpler version of a space. Put another way, quotients abstract away from some distinctions while preserving a chosen kind of structure.

The term 'quotient' has a number of mathematical definitions, each local to a specific kind of structure. For example, there are distinctive definitions for groups, vector spaces, topological spaces, metric spaces, and partial orders. The mathematical details vary from case to case, depending on the kind of object being quotiented. But the common core is an operation that involves both abstraction and structure-preservation.

In a moment, I'll explain quotients in more detail. But let me first state my answer to the ways-of-varying question:

### The Quotient Analysis

$D$  is a dimension of  $F$   $:=$   $D$ -space is a quotient of  $F$ -space.

I argued earlier that an analysis of dimensions ought to be applicable to a wide range of structures: discrete and continuous, linear and cyclical, geometric and algebraic. I've also argued that an analysis of dimensions ought to focus on the extrinsic structural relationship between dimensions and their kinds, rather than on the intrinsic structures of dimensions themselves. The quotient analysis—as we will see—satisfies both desiderata.

It's impossible, in this article, to explain the many mathematical details of quotients. But I'll focus on explaining enough for readers to understand the basic mathematical ideas behind the quotient analysis.

**A Primer on Quotients**<sup>19</sup>

Let’s start with some core concepts relevant to any quotient construction:

the <b>source space</b>	the space of F-values
the <b>quotient space</b>	the space of D-values
the <b>quotient map</b>	the structure-preserving function $q : F \rightarrow D$

A quotient space D may be thought of as a collapsed version of its source space F. The collapse is defined by the quotient map  $q$ , a surjective (though usually not injective) function from F-space to D-space. The quotient map  $q$  induces a partition over F-space, where the equivalence relation  $\sim$  is defined as follows:  $F_1 \sim F_2$  iff  $q(F_1) = q(F_2)$ . The following diagram illustrates these relationships via a quotient operation from color space to hue space:

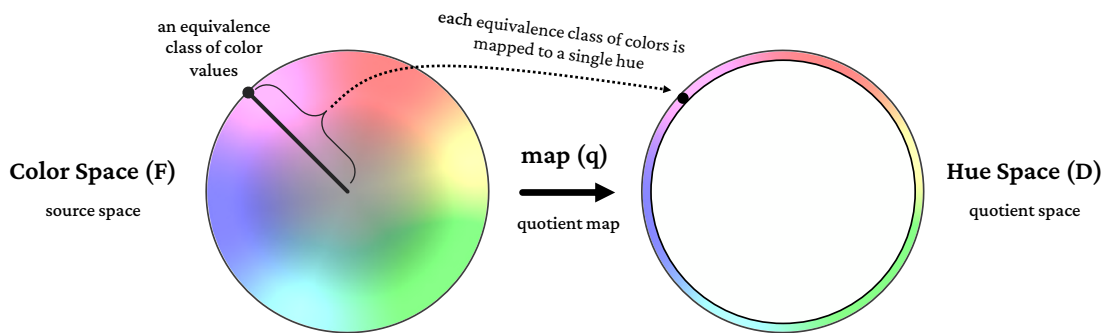


FIGURE 3: A quotient construction.

It’s worth noting that the term **quotient** is polysemous. ‘Quotient’ can mean (1) the quotient space D, (2) the quotient map  $q$ , (3) the operation of applying  $q$  to F, resulting in D, or (4) the partitioned space  $F/\sim$ . These ambiguities are mostly harmless, since there are systematic relationships between these interpretations. When it’s important to disambiguate, I’ll use ‘quotient space’, ‘quotient map’, ‘quotienting’, or ‘partitioned space’.

<sup>19</sup> Interestingly, there are two approaches to defining quotients. The first—which I explain in this section—is through structure-preserving functions, which involve mapping F-space to D-space via a surjective structure-preserving map  $q$  that captures all the structure of D-space. The second—which I explain in the APPENDIX—is through structure-inheriting partitions, which involve partitioning F-space and then inducing structure on that partition from the original F-space. These turn out to be mathematically equivalent up to isomorphism. Because of this, mathematical discussions of quotients often treat these as two perspectives on the same basic idea.

Quotients are closely related to partitions. Every quotient induces a partition. In fact, it's often useful to identify quotients by their corresponding partitions. In some contexts, quotients are even defined as partitions that inherit structure according to some principled rules.<sup>20</sup> To simplify the discussion, I'll focus here only on defining quotients in terms of structure-preserving maps (see the APPENDIX for a characterization of quotients via partitions).

Whether any arbitrary partition induces a quotient is a delicate question. The reason is that the concept of a quotient is structure-relative. When we ask whether  $D$  is a quotient of  $F$ , we need to specify the kind of structure under consideration in order to make the question precise. In the limit, if we care only about set-theoretic structure, then every partition yields a quotient (since for sets, quotients just are partitions). But for other kinds of structure—including algebraic operations, order relations, and geometries—arbitrary partitions may not preserve the relevant kind of structure.

This leads to a choicepoint for the quotient analysis. On a **restrictive version** of the view, only partitions that preserve the relevant kind of structure over  $F$ -space yield a dimension of  $F$ . On a **permissive version** of the view, every partition over  $F$ -space yields a dimension of  $F$ , though many of these partitions will relinquish certain kinds of structure.<sup>21</sup> The restrictive view makes the concept of a dimension more structurally demanding. But it also raises questions about exactly *which* kinds of structure must be preserved. It's not obvious how to answer this, since dimensions sometimes abstract away from one *kind* of structure while preserving another kind. For example, velocity has the structure of a vector space, and direction is a dimension of velocity, but direction lacks the structure of a vector space. Those skeptical that there's a principled way to define which kind of structure-preservation is needed may find themselves more drawn to the permissive view.<sup>22</sup>

To understand quotients, we need to understand the concept of a structure-preserving map. But the definition of a structure-preserving map varies across

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<sup>20</sup> There's a subtle metaphysical question about the relationship between the quotient space ( $D$ ) and the partitioned space ( $F/\sim$ ). Is each  $D$ -value literally an equivalence class of  $F$ -values? The answer turns on whether we draw distinctions between properties whose values are intensionally equivalent and whose spaces are isomorphic to each other. My official view is that dimensions are quotient spaces, but I'll leave open how these relate to partitioned spaces.

<sup>21</sup> As far as I can tell, uses of 'quotient' in mathematics underdetermine these two senses. This is because nearly all discussions of quotients occur in specific mathematical contexts, where what's of interest is whether the structure associated with that mathematical category is preserved. Even in category theory, whether an object is a coequalizer may depend on the category under which one interprets that object.

<sup>22</sup> I lean towards the permissive view, for similar reasons to why I favor universalism (about composition), plenitude (about modal profiles), and abundance (about properties).

mathematical domains. Common definitions include homomorphisms (algebraic structures), monotone maps (ordered sets), continuous maps (topological spaces), non-expansive maps (metric spaces), and linear maps (vector spaces). Because of this variance, the definition of ‘quotient’ also varies across mathematical domains. Nevertheless, these definitions are all unified by a universal definition of ‘quotient’, found in category theory.<sup>23</sup> The basic idea is that a quotient is a minimal way of collapsing a pair of structure-preserving maps.

A **quotient map** is a special kind of structure-preserving map. The intuitive idea is that a quotient map  $q : F \rightarrow D$  must not only be structure-preserving, but must also capture all of the codomain’s elements (meaning the function is *surjective*) and relations (meaning the function is *final*—there is no additional structure in the codomain beyond those captured by the domain).<sup>24</sup> Put another way, every  $D$ -value must be the image of an  $F$ -value under  $q$ , and  $D$ -space must have no additional structure beyond that which is preserved under  $q$ . A quotient space can then be defined as follows:

$D$ -space is a **quotient** of  $F$ -space  $\stackrel{=_{def}}{}$  there exists a surjective, final structure-preserving map  $q : F \rightarrow D$

This primer—though brief—is enough to cover the core mathematical ideas. But to capture the whole story, I need to also say a few things about how the mathematics connects to the metaphysics.

### From the Mathematics to the Metaphysics

There’s an elegant way to bridge the mathematics with the metaphysics. The key insight comes from thinking about **relations**: a concept that will be familiar to both mathematicians and metaphysicians.

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<sup>23</sup> The relevant concept is that of a ‘coequalizer’, which generalizes the definitions of ‘quotient’ in specific mathematical categories (sets, vector spaces, topological spaces, rings, etc.). More technically, a *coequalizer* is a morphism  $q : Y \rightarrow Q$  that’s universal with respect to the following property: for some pair of morphisms  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$ ,  $q \circ f = q \circ g$ . The intuitive idea is that a coequalizer of morphisms  $f$  and  $g$  is the minimal way of collapsing together  $f$  and  $g$ . See Mac Lane [1998] for a classic text on category theory.

<sup>24</sup> To make this precise, we need to also assign each space a *signature*, an ordered tuple of the structure (relations, operations, functions, etc.) on the set. This allows us to define which structure in the domain corresponds to which structure in the codomain. For example, the signature enables each  $F$ -relation to be matched with a corresponding  $D$ -relation.

Every property space is structured by relations, such as relations of similarity and magnitude.<sup>25</sup> For example, the geometry of color space comes from the similarity relations between color values. Furthermore, both saturation values and brightness values stand in magnitude relations, which generates the ordinal structure of those spaces. And color qualities may also stand in other relations, such as vivacity or precision.<sup>26</sup>

This observation allows us to think about any property space as a **relational space**, meaning a set of elements endowed with some relations. The elements are the values of the property, and the relations are the kinds of higher-order relations expressed above. For relational spaces, structure-preserving maps are **relational homomorphisms**, meaning functions where relations in the domain are mirrored by relations in the codomain.<sup>27</sup> More specifically, a map  $q$  is a relational homomorphism just in case whenever a relation holds between some values  $F_1, \dots, F_n$ , the corresponding relation holds between the images of those  $F$ -values under  $q$ . When this condition is satisfied, the map  $q$  “preserves the relations” over  $F$ -space.

Relational spaces are one of the most general ways of thinking about mathematical spaces. In fact, any mathematical space can, in principle, be characterized as a relational space.<sup>28</sup> And in philosophical work on “structuralism,” structure is often characterized in terms of relations. Because of this, it’s reasonable to think that the mathematical category of relational spaces is the kind of mathematical structure most relevant for the metaphysics of property spaces.

When mathematicians think about relational spaces, however, they aren’t thinking about relations in metaphysical terms. In non-mathematical contexts, relations are polyadic properties: for example, ‘taller than’, ‘more similar to’, and ‘between’ all express relations. By contrast, the relations that occur in mathematical contexts are purely

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<sup>25</sup> The limit case concerns property spaces that have merely set-theoretic structure. But even in those cases, we can define the set of relations over that space as the empty set.

<sup>26</sup> See Lee [2021] on a formal framework for modeling precision in quality-spaces.

<sup>27</sup> More technically,  $q : F \rightarrow D$  is a *homomorphism* iff for any relations  $R$  (over  $F$ -space) and  $R'$  (over  $D$ -space),  $R(F_1, \dots, F_n) \rightarrow R'(q(F_1), \dots, q(F_n))$ .

<sup>28</sup> Usually, the relations that characterize relational structures are understood as on/off entities: for any  $n$ -adic relation and any  $n$ -tuple of individuals, either those individuals instantiate the relation or not. This leads to puzzles about how to relationalize geometrical structures, where there’s a multitude of distinct relations that are all structurally related. For example, in a metric space, there may be infinitely many distinct distances, each of which will be associated with a distinct relation. I suspect that the solution is to think of relations—like any other property—as associated with spaces of values. For example, just as color is a property with many values (specific colors), distance is a relation with many values (specific distances). This approach to relational structures makes them a more powerful tool for uniting various kinds of mathematical structures. For limits of space, however, I won’t develop this idea in detail.

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mathematical in nature. As an analogy, think about how points in mathematical spaces needn't be interpreted as any particular kinds of objects. Instead, points are dummy objects, individuated purely by the structural roles that they play.

If we understand quotients in a purely mathematical way, however, then we run into a problem with the metaphysics. The problem is this: any properties  $F$  and  $G$  will, from a purely mathematical point of view, stand in exactly the same quotient relations. Suppose, for example, that both duration and length have the structure of  $\mathbb{R}^+$ . Then the space of duration values is isomorphic to the space of length values. But we obviously wouldn't want these properties to stand in exactly the same dimension relations.

The solution is to think of the relations as interpreted, rather than as purely formal. In other words, the relevant relations are the metaphysical relations that structure  $F$ -space and  $D$ -space, rather than purely mathematical relations that characterize the corresponding purely mathematical spaces. This ensures that isomorphic properties won't stand in exactly the same dimension relations. Whereas mathematicians care only about distinctions up to isomorphism, metaphysicians care also about the natures of the target elements and relations. By appealing to metaphysical relations between properties—such as those associated with similarity, magnitude, and precision—we can interpret quotients as capturing metaphysical (rather than only mathematical) structure.

You might wonder which metaphysical relations are the ones that need to be preserved. That's a deep and important question. But it's also a question that turns on more general issues that lie beyond the scope of this paper.<sup>29</sup> A starting point, for this paper, is that properties are associated with spaces of values, and that we have some knowledge of the structures of these spaces. For example, it's a reasonable datum that we know that color space has a certain geometric structure, and that velocity space has a certain vector space structure. The metaphysical relations discussed in this section are whichever relations structure property spaces.

The move developed above enables the quotient analysis to be sensitive to the nature—and not merely the structures—of the target properties. This enables us to recover the right dimension relations between properties. Furthermore, this allows us to better capture the sense in which quotients involve abstracting away from a structure in order to generate a simpler version of that very same structure.

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<sup>29</sup> I suspect a complete answer will require appealing to both measurement theory (to explain how to associate properties with mathematical structures) and metasemantics (to explain how we can know that the properties we denote actually have the structures we ascribe to them).

### Applying the Analysis

To apply these ideas to a concrete example, let's consider how the quotient analysis works in the case of hue and color. Let  $q$  be a map from color space ( $F$ ) to hue space ( $D$ ), where each color value is mapped to its corresponding hue value. Intuitively, this map preserves the structure of hue while abstracting away from saturation and brightness. But does it satisfy the definition of a quotient?

Since every hue value will be the image of an equivalence class of color values (namely, all those that entail that hue value), it's obvious that  $q$  is surjective. Furthermore, the relations that structure hue space will always be applicable to the corresponding color values. For example, hue values stand in similarity relations that are naturally modeled by angular distance, measured by how far apart two hue values are along the hue circle. And these angular distance relations apply just as well to color values. Furthermore, whenever color values  $F_1$  and  $F_2$  stand in some angular distance relation, their images  $D_1$  and  $D_2$  will likewise stand in those same relations. In more intuitive terms, the similarity relations that capture hue space are mirrored by similarity relations in color space. These observations illustrate how the map  $q$  will be surjective, final, and structure-preserving. From this, we can conclude that  $q$  is a quotient map from color space to hue space.

It's easy to see that other examples of dimension and their kinds will follow a similar pattern. For example, consider speed and velocity. Suppose we partition the space of velocity by speed, meaning we equate all velocity values that are the same in speed. Then the partitioned space preserves the magnitude structure of speed while abstracting away from the angular structure of direction. Likewise, the map  $q$  from velocity values to speed values preserves magnitude relations while abstracting away from directional relations. Therefore, speed is a quotient of velocity.

In general, the quotient analysis of dimensions says that for  $D$  to be a dimension of  $F$  is for there to be a final surjective structure-preserving map  $q$  from  $F$ -space to  $D$ -space. But the relevant notion of structure-preservation concerns not only purely mathematical relations that characterize the mathematical structure of the relevant spaces, but also the metaphysical relations (such as similarity and magnitude) that define these property spaces.

### Dimensions as Quotients

You might wonder, at this point, what a counterexample to the quotient analysis would look like. A natural strategy is to find a case where some  $D$  is intuitively a dimension of some  $F$ , yet where the space of  $D$ -values *cannot* be recovered by quotienting the space of  $F$ -values. This would require  $D$  to be structurally richer than  $F$ , in some respect. But in

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every intuitive example where some  $D$  is intuitively a dimension of some  $F$ , the structure of  $D$ -space is more coarse-grained than the structure of  $F$ -space. If this pattern is invariant across every example of a dimension, then it ought to be captured by an analysis of the dimension relation. The quotient analysis captures the pattern.

You might also wonder whether there are examples where  $D$ -space is a quotient of  $F$ -space yet where  $D$  isn't obviously a dimension of  $F$ .<sup>30</sup> I'm open, in principle, to a theory of dimensions that imposes additional restrictions. But the challenge is to find constraints that (i) apply across a comparably wide variety of mathematical structures: algebraic and geometric, discrete and continuous, linear and cyclical, and (ii) analyze the dimension relation in terms of a natural structural kind. Unless there's a plausible restriction that retains the generality and naturalness that makes quotients attractive in the first place, I'm inclined to stand by the quotient analysis.

It's worth noting that the quotient analysis doesn't appeal to notions such as essence, grounding, or naturalness.<sup>31</sup> However, it also doesn't preclude an appeal to such notions to develop more metaphysically robust versions of the quotient analysis. For example, one might think that the structural relationship captured by the quotient analysis must hold in virtue of the essences of  $D$  and  $F$ .

As with the partition analysis, it's easy to verify that each of the structural principles expressed earlier—UNIVERSALITY, IMPOSSIBILITY, VARIABILITY, MULTIPLICITY, and ASPECTUALITY—can be satisfied. The first three principles follow straightforwardly from the quotient analysis (in analogous ways to the partition analysis). The last two principles turn on whether we permit the trivially coarse and trivially fine quotients—a point I'll return to in the next section. In my mind, it's remarkable that a universal mathematical concept can account for each of these structural principles. After all, the principles were developed as intuitive criteria that served as desiderata for a theory of dimensions. The fact that each criterion can be recovered by the quotient analysis illustrates the elegance of the view.

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<sup>30</sup> One class of cases is where  $F$  is one-dimensional yet permits non-trivial quotients. For example, consider electric charge, which is often taken to have the structure of  $\mathbb{R}$ . It's possible to quotient  $\mathbb{R}$  into three equivalence classes, which contain (1) all negative numbers, (2) zero, and (3) all positive numbers. The result is a quotient that inherits  $\mathbb{R}$ 's order structure (since positive > zero > negative). But that quotient still strikes me as a reasonable example of a dimension: it's natural to call polarity (positive vs. zero vs. negative) a dimension of charge.

<sup>31</sup> I suspect there's no universal answer as to whether dimensions or their kinds are more fundamental. Given this, I think an analysis of dimensions ought to let us to settle those questions on a case-by-case basis.

The key virtue of the quotient analysis (over the partition analysis) is that it captures the ways in which the structures of dimensions are systematically connected to the structures of their kinds. The quotient analysis captures these connections using a universal mathematical concept.

I began this paper by noting that dimensions are—roughly—ways of varying with respect to a kind. The quotient analysis allows us to take this rough idea and make it formally precise.

## §5 The Dimension Relation

Suppose we accept the quotient analysis. We can now explore some of the formal properties of the dimension relation. To start, I want to consider two edge cases of dimensions, corresponding to the two trivial kinds of quotients. These can be defined by their corresponding partitions:

**The Identity Quotient:**  $\forall F_i \forall F_j: F_i \sim F_j \text{ iff } F_i = F_j$   
**The Singleton Quotient:**  $\forall F_i \forall F_j: F_i \sim F_j$

Hence, the **identity quotient** (the trivially fine quotient) equates no distinct F-values, meaning that every F-value is its own equivalence class. Here the resultant quotient space is F-space itself. On the other hand, the **singleton quotient** (the trivially coarse quotient) equates every F-value, meaning there’s only one equivalence class over the whole space. Here the resultant quotient space corresponds to a maximally determinate property that’s instantiated just in case any F-value is instantiated. We can likewise define the corresponding “dimensions”:

D is an **identity dimension** of F  $=_{def}$  D-space is an identity quotient of F-space  
 D is a **singleton dimension** of F  $=_{def}$  D-space is a singleton quotient of F-space

Should we include these in a theory of the dimension relation? The choicepoint corresponds to the question of whether to accept ASPECTUALITY and MULTIPLICITY. To accept ASPECTUALITY is to reject identity quotients (since those generate a quotient space that’s equivalent to the original space).<sup>32</sup> To accept MULTIPLICITY is to reject universal quotients (since those generate a quotient space with just a single value).

<sup>32</sup> Actually, this turns on some further questions about property individuation. But I’ll assume that if F and G are both intensionally equivalent and structurally isomorphic, then F = G. If we accept this, then

To some extent, this is a verbal question, rather than a substantive metaphysical stance. After all, no matter which we take to be the “official” definition, we can still draw the following distinctions:

D is a **proper dimension** of F  $\stackrel{=_{def}}{}$  D is a non-trivial quotient of F  
 D is an **improper dimension** of F  $\stackrel{=_{def}}{}$  D is a (possibly trivial) quotient of F

It may initially feel odd to think of the edge cases—involving identity quotients and singleton quotients—as counting as dimensions. But think about how it likewise feels odd when one first learns to think of the parthood and subset relations as reflexive. The motivations for the improper definitions of ‘part’ and ‘subset’ don’t come from our ordinary judgments. Instead, the motivation is that those definitions yield more elegant formal frameworks. In my view, the question of how to best formalize the dimension relation ought to be treated analogously.

It turns out, in fact, that the improper definition yields a more elegant formal framework for dimensions. To start, the improper definition makes the dimension relation reflexive, transitive, and anti-symmetric: the properties definitive of a partial order. **Reflexivity** holds because of the identity quotient. **Transitivity** holds because any composition of quotients is itself a quotient. **Anti-Symmetry** is more delicate: it requires a moderately coarse view of properties. But if we accept that properties that are both intensionally equivalent and structurally isomorphic are identical, then quotient symmetries will entail identity.

In fact, on the improper definition of ‘dimension’, every property F is associated with a **dimensional lattice**. A *lattice* is a partial order where every pair of elements has a unique supremum (least upper bound) and unique infimum (greatest lower bound). The elements of the lattice are dimensions of F. Equivalently, the elements can be thought of as quotients over F-space. The order structure is defined by the dimension relation: in particular,  $D \leq F \stackrel{=_{def}}{}$  D is a dimension of F. The supremum is the identity quotient, and the infimum the singleton quotient. Here’s a diagram of a very small lattice:

proper dimensions can be defined as those that respect ASPECTUALITY and MULTIPLICITY, while improper dimensions include also dimensions that violate those principles.

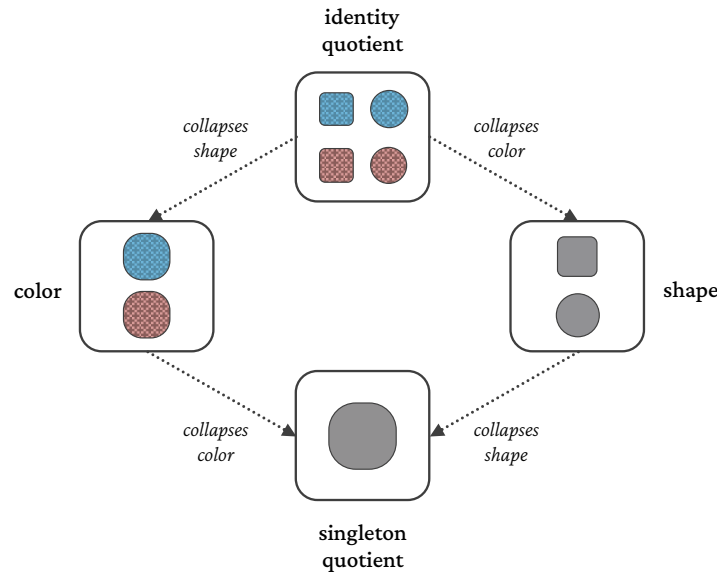


FIGURE 4: A dimensional lattice.

This yields an elegant hierarchy of dimensions. Every property is associated with a hierarchy of dimensions, which come in varying levels of fineness and coarseness. However, this analysis might also invite a worry—which I turn to next—about the relationship between the quotient analysis of dimensions and the notion of dimensionality.

**An Inequality**

I previously distinguished the following two questions:

- ways-of-varying?**      What is it for some  $D$  to be a dimension of some  $F$ ?
- $n$ -dimensionality?**      What is it for some  $F$  to be  $n$ -dimensional?

The quotient analysis is an answer to the ways-of-varying question. But you might worry that this answer has unpalatable implications for the  $n$ -dimensionality question. For example, color space is three-dimensional. But there are many ways to quotient color space. In general, the quotient analysis entails the following inequality:

INEQUALITY

$$\text{dimensionality}(F) \neq \text{number of } D\text{'s such that } D \text{ is a dimension of } F$$

This may initially seem surprising. But in my view, it's a result we ought to embrace (even if we were to set aside the quotient analysis). A dimension, at first pass, is a way of varying with respect to a kind. But there are many cases where there are more than  $n$  ways of varying *wrt* an  $n$ -dimensional property. For example, it's hard to deny that both hue and redness are dimensions of color. But once we permit both of those, we ought to also permit saturation and brightness (since those accompany hue) and greenness, blueness, blackness, whiteness, and so forth (since those arguably stand or fall with redness). This means there are many properties—much more than three—that stand in the dimension relation to color. Yet it's uncontroversial that color is three-dimensional. The best resolution, in my view, is to accept INEQUALITY.

This doesn't mean that there's no connection between dimensionality and the dimension relation. The connection, in my view, can be clarified by thinking about **dimensionalization**, meaning the assignment of dimensions to a property. For example, hue, saturation, and brightness is one way of dimensionalizing color, while red/green, blue/yellow, black/white is another way of dimensionalizing color. While there are countless properties that are dimensions of color, any adequate dimensionalization of color will assign exactly three properties.

I explore questions about dimensionalization in more detail in other work. But here's a brief overview of the basic idea. To dimensionalize a property  $F$  is to take a step in constructing a model of  $F$ , meaning a mathematical representation of the space of  $F$ -values. The best models of  $F$  are those that optimize the balance between simplicity (roughly, number of dimensions postulated by the model) and strength (roughly, proportion of  $F$ -differences captured by the model). Since optimal models of color always postulate three dimensions, color is three-dimensional. But since there are many ways of quotienting color space, there are countless properties that stand in the dimension relation to color. Any of these properties may be selected when dimensionalizing color, though only collections of dimensions that maximize the ideal of orthogonality (defined in terms of degree of free recombability between values) will be adequate dimensionalization of color.

## §6 Dimensions vs. Parts

Here's an intriguing—though speculative—idea: the dimension relation is the “metaphysical dual” to the parthood relation. Each is a basic way of extracting something simple from something complex.

This idea can be made more precise by thinking about the mathematical structures associated with each relation. I'll argue that this metaphysical duality—between dimensions and parts—is justified by a mathematical duality—between partitions (or

quotients) and subsets (or subobjects). The picture that results generates a pleasing symmetry between these metaphysical concepts.

### Partitions vs. Subsets

To illustrate, think about sets. There are two ways to “simplify” a set. The first is by taking a **subset**. This cuts away a portion without collapsing any distinctions. The second is by taking a **partition**. This collapses some distinctions without cutting any portion.

Of course, there’s a close relationship between these operations. Any subset canonically induces a partition: namely, a binary partition that divides a set into the subset and its complement. Any partition canonically induces a set of subsets, where each subset corresponds to a cell of the partition. But this merely illustrates the kind of mathematical duality I have in mind.

It turns out that this mathematical relationship runs much deeper. A partition, as I’ve explained, is closely related to a quotient. In fact, it’s common to think of quotients as partitions that inherit structure from their source spaces. Therefore, dimensions are associated with partitions, while parts are associated with subsets. But quotients, as I’ve noted, are basically partitions with structure-inheritance. You might then wonder: is there a mathematical concept that stands to subsets as quotients stand to partitions?

<b>partitions</b>	:	<b>quotients</b>
<b>subsets</b>	:	?

Before I answer, let me raise another question. In mathematics, a concept that recurs across different contexts is **duality**. As examples, consider the modal operators  $\Box$  and  $\Diamond$ , or the logical operators  $\wedge$  and  $\vee$ , or injections vs. surjections, or infimums vs. supremums. The basic idea is that duality arises when there’s a systematic correspondence between structural roles. In category theory, the notion of duality can likewise be generalized, where this means (roughly) reversing the directions of the arrows in a definition. This definition of duality can recover the duality in local contexts, such as the ones expressed above. Now, I also mentioned earlier that the concept of a quotient has a universal definition, expressible in the language of category theory. This leads another question: is there a mathematical dual to the concept of a quotient?

The answer—to both questions—is a **subobject**. This means—roughly—an object that’s contained within another object. This includes subsets (in set theory), subgroups (in group theory), subspaces (in topology), subposets (in order theory), and

subgraphs (in graph theory). Just as quotients are basically partitions with structure-inheritance, subobjects are basically subsets with structure-inheritance.

The duality between quotients and subobjects can also be illustrated by thinking about the associated mappings. A *quotient map*, as noted previously, is a structure-preserving map that's both *surjective* (all elements in the codomain are mapped to) and *final* (there is no additional structure in the codomain). A related concept, when thinking about subobjects, is an *embedding*. This can be defined as a structure-preserving map that's both *injective* (distinct inputs map to distinct outputs) and *initial* (there is no additional structure in the domain). The following diagram depicts the difference between embeddings and quotients:

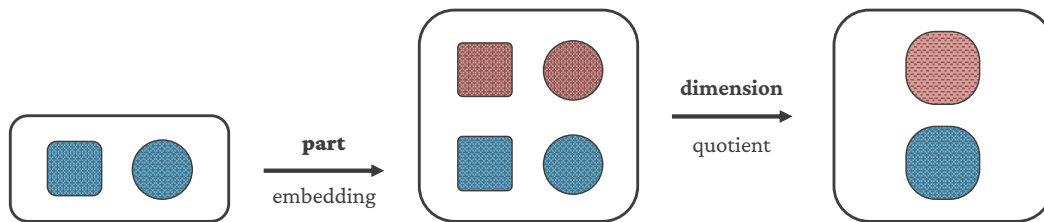


FIGURE 5: Embeddings vs. Quotients.

For limits of space, I won't go into the details of subobjects in this paper. But the analogies—as well as the explanation of how quotients relate to partitions—ought to be enough to see why this indicates a metaphysical duality between dimensions and parts.

Therefore, dimensions stand to partitions stand to quotients as parts stand to subsets stand to subobjects. This generates a pleasing symmetry between dimensions and parts, and an elegant way of connecting both of these basic metaphysical relations to general mathematical kinds.<sup>33</sup>

### Dimensions vs. Parts

The analogies between dimension and parts can be developed further. When thinking about *dimensions*, it's important to think also about *kinds* and *values*. What are the corresponding concepts for parthood? Here's a hypothesis:

<sup>33</sup> The ideas here might be compared to Koslicki [2008], who develops a structured analysis of the parthood relation. On her view, objects are structured wholes whose parts must be arranged in a certain way. The mathematical notion of a subobject might be seen as a formal counterpart to this idea.

**dimensions**     :     **kinds**     :     **values**  
**parts**            :     **wholes**    :     **atoms**

The dimension relation relates dimensions and kinds; the parthood relation relates parts and wholes. Every property comes with a space of values (the basic units of properties that have no further determinates); every particular comes with a set of atoms (the basic units of particulars that have no further parts). For any kind  $F$ , we can define an associated domain of  $F$ -values; for any whole  $x$ , we can define an associated domain of  $x$ -atoms.

To make the comparison fully analogous, we would need to understand *atoms* in a relativized way, where atoms are relativized to wholes. Early on, I drew a distinction between values and realizers, where the values of a property can themselves be multiply realized. Similarly, instead of understanding an  $x$ -atom as an atom simpliciter, it might instead be understood as an atom relative to  $x$ . Just as two objects can differ microphysically without differing *wrt*  $F$ , two objects can differ microphysically without differing *wrt*  $x$ .<sup>34</sup>

The table below depicts all the relationships I’ve discussed (dimension terms are in blue, parthood terms are in red, and neutral terms are in black):

	<i>metaphysical relation</i>		<i>unstructured mathematical relation</i>		<i>structured mathematical relation</i>	
<i>aspect</i>	dimension	part	partition	subset	quotient	subobject
<i>complex</i>	kind	whole	set		space	
<i>unit</i>	value	atom	element		element	

FIGURE 6: Dimensions vs. Parts.

These remarks express some initial comparisons between dimensions and parts. But there’s much more to explore here. If my hypothesis is correct—if there are indeed these dualities within the metaphysics and the mathematics—then there’s a whole research program to be developed on these connections between the mathematics and the metaphysics.

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<sup>34</sup> This isn’t the standard way of understanding a mereological atom. But it strikes me as a sensible and underexplored way of thinking, and draws a closer analogy between dimensions and parts.

## §7 Multiplicity Structures

Multidimensionality is sometimes conflated with other kinds of “multiplicity structures.” Here’s a brief list of definitions:

F is <b>multidimensional</b>	= <sub>def</sub>	there are multiple ways of varying <i>wrt</i> F.
F- <b>pluralism</b>	= <sub>def</sub>	there are multiple kinds denoted by ‘F’.
F is <b>determinable</b>	= <sub>def</sub>	there are multiple F-values.
F is a <b>genus</b>	= <sub>def</sub>	there are multiple species of F.

These are all distinct.<sup>35</sup> In what follows, I’ll briefly clarify the relationship between multidimensionality and the other kinds of structures. In the next section, I’ll discuss a more subtle distinction between dimensions and “parameters.”

### Pluralism

F-pluralism	≠	F is multidimensional
F is multidimensional	≠	F-pluralism

Just because there are multiple kinds of Fs doesn’t mean there are multiple ways in which Fs can vary from each other. As an example, everyone agrees that velocity is multidimensional (speed and direction), but nobody thinks that there are multiple kinds of velocity. Hence, multidimensionality doesn’t entail pluralism. As another example, almost nobody who endorses pluralism about truth thinks that truth is multidimensional (instead, even pluralists about truth tend to think that there are only two truth-values).

If F is multidimensional, then (by UNIVERSALITY) anything with an F-value has a value along each of the dimensions of F. But someone who endorses pluralism about F doesn’t need to think that every instance of F is an instance of each kind of F. For example, one might be a value pluralist, in the sense that one thinks that multiple kinds of things (say, pleasure and knowledge) are valuable. But that doesn’t mean that each instance of value has some degree of pleasure and some degree of knowledge.

<sup>35</sup> But these structures are also sometimes conflated. For example, Hedden & Munoz [2024] seem to equate value pluralism, value multidimensionality, and value multiparametricity.

**Determinability**

F is determinable	≠	F is multidimensional
F is multidimensional	⊨	F is determinable
F is determinate	⊨	F is zero-dimensional

A *determinable* is a property that has multiple values, or *determinates*.<sup>36</sup> For example, color is a determinable, and each specific color—red<sub>34</sub>, green<sub>17</sub>, etc.—is a determinate of color. If there are multiple ways of varying with respect to F, there must be at the very least two distinct values of F,<sup>37</sup> which entails that F is determinable. But F can be determinable without being multidimensional. For example, mass is determinable (there are many mass values) but unidimensional. Furthermore, questions about determinability underdetermine questions about multidimensionality. For example, the mere fact that color is a determinable (with each color value as a determinate) leaves open the dimensionality of color. In order to extract the dimensionality of color, we would need to know not only the determinates of color, but also the structural relations between those determinates.

On the other hand, any property that’s determinate has, by definition, only one value. But single-valued properties just are zero-dimensional properties. Hence, there’s a close connection between being superdeterminates and zero-dimensionality. You might object by noting that some zero-dimensional properties—say, being a table—don’t seem to be superdeterminate. But this objection conflates a property having a single value with a property having a single realizer. Nearly every property is multiply realizable, in that there are distinct states of affairs in virtue of which that property is instantiated. But not all those properties are multivalued: it’s not the case that for any multiply realizable F, it’s possible for *x* and *y* to vary wrt F.

You might wonder whether we could understand dimensions of F as (non-maximal) determinates of F. For example, redness is a dimension of color, and redness is also a determinate of color. But this trades upon two distinct interpretations of ‘redness’: (1)

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<sup>36</sup> See Wilson [2023].

<sup>37</sup> Why two values? Well, if F is multidimensional, then F must have at least two proper dimensions. Suppose, then, that D<sub>1</sub> and D<sub>2</sub> are both proper dimensions of F. Then D<sub>1</sub> and D<sub>2</sub> must each have at least two values. Now, if D<sub>1</sub> and D<sub>2</sub> each have two values, then it’s natural to think that there must be at least four possible values of F (since 2 × 2 = 4). But since dimensions needn’t be orthogonal, it’s possible that both (1) the first value of D<sub>1</sub> is impossible with the first value of D<sub>2</sub>, and (2) the second value of D<sub>2</sub> is impossible with the second value of D<sub>1</sub>. That rules out two possibilities amongst the four permutations of D<sub>1</sub> and D<sub>2</sub> values, leaving two remaining possibilities. Hence, it’s possible for F to be multidimensional while having only two values.

as a way of varying *wrt* color, vs. (2) as a region of color space. Consider, for comparison, hue (which doesn't admit of the same ambiguity). There's no sense in which hue is a determinate of color, since there's no subregion of color space that we could identify with hue. Furthermore, there's no subregion of color space that is devoid of hue values, since every color value is associated with hue values. Hence, while both dimensions and determinates are properties, it's never the case that a dimension of F can be identified with a determinate of F.

**Genus/Species**

s is a species of G             $\models$      $\neg$ (s is a dimension of G)  
 D is a dimension of F         $\models$      $\neg$ (D is a species of F)

In other words, the dimension relation is mutually exclusive from the 'species of' relation. To elicit some initial intuitions, consider how human is a species of (but not a dimension of) mammal, and how hue is a dimension of (but not a species of) color.

The reason is that distinct species of a genus exclude each other (human and walrus are both species of the genus mammal, but nothing is both a human and a walrus), while distinct dimensions of a property must be co-instantiable (since both hue and saturation are dimensions of color, anything that is colored must have *both* a hue value and a saturation value). More abstractly, if both B and C are species of A, then B and C must have disjoint extensions. By contrast, if both D<sub>1</sub> and D<sub>2</sub> are dimensions of F, then every individual with an F-value has both a D<sub>1</sub>-value and a D<sub>2</sub>-value.

There are also other differences between the genus/species relation and multidimensionality. Usually, both genres and species are denoted using sortal terms, which can be combined with determiners ('the F', 'an F', etc.). For example, 'the mammal' or 'the walrus' is used to denote individual entities that fall under the category mammal or walrus. By contrast, we rarely use determiners to denote individuals that have values with respect to dimensions. For example, 'the hue' doesn't denote individual entities that have hue values, but instead hue values themselves.

**§8    Dimensions vs. Parameters**

I want to close by drawing an important but underexplored distinction between dimensions and "parameters." Let me start with an observation:

a puzzling  
 observation

Some philosophers claim that moral value is multidimensional *because* it's a function of welfare and equality. Yet moral value, on such views, is often represented using scalars (the elements of a

one-dimensional scale). The idea is that the dimensions of value (welfare and equality) aggregate to yield overall value (measured by scalars). But if value is best represented using scalars, then how could it also be multidimensional?

The puzzle is general. There are some  $F$ s where it's claimed both that (1)  $F$  is multidimensional, and (2)  $F$ -space has the structure of a one-dimensional scale. But these claims seem straightforwardly contradictory.

To resolve the puzzle, we need to distinguish the concept of a dimension from the concept of a *parameter*. Drawing this distinction will not only clarify a verbal confusion, but will also enable us to identify a systematic structural relationship between two structural concepts. In turn, this will have consequences for philosophical debates about aggregation and comparability.

### Dimensions vs. Parameters

In mathematics, an argument of a function is sometimes called a **parameter**. A function with multiple parameters is thus a function that takes in multiple inputs. When thinking about mathematical functions, there's little risk of confusion between (1) the number of parameters of a function, and (2) the dimensionality of its codomain. For example, a multiparametric function—such as  $f(x, y) = x + y$ —might have  $\mathbb{R}$  as its codomain, and a uniparametric function—such as  $f(x) = (x, x)$ —might have  $\mathbb{R}^2$  as its codomain.

In philosophy, the term 'parameter' doesn't have a standard meaning. But it's worth coining the term, because there's a philosophical concept that plays an analogous role to the mathematical concept. Sometimes, values along a property  $F$  are a function of values along multiple parameters. Furthermore, a number of philosophical discussions have used the term 'dimension' to express this concept.<sup>38</sup> But it's important to disentangle dimensions from parameters:

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<sup>38</sup> For some examples, see Hedden & Munoz [2024], D'Ambrosio & Hedden [2024], Hedden & Nebel [2024], and D'Ambrosio & Stoljar [ms]. In each of these works, the term 'dimension' is used to express what I'm calling a parameter: namely, an input to the function that determines overall  $F$ -values. For example, Hedden & Muñoz [2024] define dimensions of value as "distinct respects in which a thing can be good or bad, to which overall value is responsive," and D'Ambrosio & Hedden [2024] treat the "dimensions" of a multidimensional adjective  $F$  as inputs to an aggregation function that yields overall  $F$ -ness verdicts. In the terminology I'll later introduce, these are parameters of  $F^\circ$  rather than dimensions of  $F$ . My arguments in this section aren't intended to undermine the ideas in these papers, but instead to draw a distinction that clarifies the target subject-matter.

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<p>D is a <b>dimension</b> of F</p>	<p>≈</p>	<p>D is a way of varying <i>wrt</i> F.</p>
	<p>:=</p>	<p>D-space is a quotient of F-space.</p>
<p>P is a <b>parameter</b> of F</p>	<p>≈</p>	<p>P-values are inputs to the function that determines F-values.<sup>39</sup></p>

If moral value is determined by aggregating welfare and equality, then welfare and equality are each parameters of moral value. But from that, it doesn't follow that either welfare or equality are *dimensions* of moral value. In fact, if we grant that moral value is best represented using a one-dimensional scale, it's reasonable to infer that *neither* welfare nor equality are dimensions of moral value.

There are many other examples that disentangle dimensions from parameters. Consider: (a) density is scalar, yet a function of mass and volume, (b) expected utility is scalar, yet a function of utility and probability, and (c) GPA is scalar, yet a function of grades in each of one's courses. In each case, the target kind is typically represented using a subset of  $\mathbb{R}$ . That's evidence that the kind itself is one-dimensional, even though values along that kind are determined by multiple parameters.<sup>40</sup>

To evaluate whether a property is a parameter (and not a dimension) of F, it's useful to appeal to VARIABILITY: if D is a dimension of F, then D-differences are F-differences. Consider: two objects *x* and *y* can differ in both mass and in volume without differing in density. As long as the ratio of mass to volume is the same, *x* and *y* have the same density. Therefore, neither mass nor volume satisfy VARIABILITY *wrt* density. This indicates mass and volume aren't dimensions of density. But since density *just is* mass over volume, mass and volume are parameters of density. Analogous considerations apply to the other examples mentioned above.

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<sup>39</sup> I've purposefully left open the metaphysical relation connecting parameters to their kinds. One option is to understand determination in modal terms. But another option is to appeal to a stronger relation, such as grounding. This strikes me as an attractive way to precisify the notion of a parameter, but I'll leave open which analysis is best. My main goal is not to develop a final analysis of parameters, but instead to explain how parameters are conceptually distinct from dimensions.

<sup>40</sup> What's the relationship between parameters and the sense of 'dimension' at stake in dimensional analysis? Well, the latter sense of 'dimension' is usually defined as a property of quantities, but parameters needn't be restricted to quantities. Also, the dimension of a quantity (in dimensional analysis) is expressed as a product of powers, but the parameters of F needn't be related to each other in such a way. Nevertheless, perhaps the notion of a parameter can generalize the sense of 'dimension' at stake in dimensional analysis.

### A Structural Analysis of Parameters

There are some systematic structural relationships between dimensions and parameters. But to identify those connections, we need to first make the notion of a parameter more precise. To do that, we might adopt the same approach that we took earlier for dimensions, where we identify some structural principles that relate parameters and their kinds.

It turns out—remarkably—that a natural way of precisifying the notion of a parameter comes from *inverting* the structural principles that we defined earlier for dimensions. The inversion comes from swapping the role of the kind with the role of the dimensions/parameters. Furthermore, we need to not only swap dimensions with parameters, but we need to also formulate the principles as concerning a complete set of parameters (rather than just a single parameter) for the target kind  $F$ .<sup>41</sup> Let  $P = \{P_1, P_2, \dots, P_n\}$  be such a set, where each  $P$ -value is itself a specification of values along each  $P_i$ . Then consider the following:

UNIVERSALITY*	Everything with a $P$ -value has an $F$ -value.
INCOMPOSSIBILITY*	Nothing with a single $P$ -value has multiple $F$ -values.
VARIABILITY*	All $F$ -differences are $P$ -differences.
ASPECTUALITY*	Not all $P$ -differences are $F$ -differences.
MULTIPLICITY*	Some things with $P$ -values differ in $F$ -values.

There's a pleasing symmetry when we compare these principles to the earlier principles characterizing dimensions. In my view, the systematicity of this transformation is evidence that we have found another structural duality. That is, there's one way in which dimensions and parts are duals (quotients and subobjects are mathematical duals), but another way in which dimensions and parameters are duals (through the dual definitions of these structural principles).<sup>42</sup> This is also evidence that questions about parameters may be as rich as questions about dimensions (and parts).

<sup>41</sup> What does 'complete' mean? Well, one answer is that a complete set includes *all* the parameters of  $F$ . But that might be overkill, since it might be the case that many parameters overlap (and so including all will be massively redundant). A more minimal option is to define a set of parameters  $P_F$  as complete relative to  $F$  just in case each  $P_F$ -value uniquely determines an  $F$ -value.

<sup>42</sup> Does this mean that the concept of duality is itself polysemous? That's an interesting question, and the answer isn't obvious. In the present context, however, it seems to me that there's a common core when thinking about parts as duals vs. parameters as duals. My analysis of dimensions involves both a mathematical operation (quotients) and a set of propositions involving quantifiers and modal operators (the structural principles). The dual of quotients yields the mathematical analogue of parthood; the collection of the duals of the principles yields the structural principles for parameters.

Just as we drew a distinction between proper vs. improper dimensions, we can draw an analogous distinction between proper vs. improper parameters. The distinction, like before, turns on whether we require or relinquish ASPECTUALITY\* and MULTIPLICITY\*. In what follows, I'll focus only on *proper parameters* and *proper dimensions* (omitting 'proper' in the prose).

Proper dimensions and proper parameters are mutually exclusive. More precisely, if  $D$  is a proper dimension of  $F$ , then  $D$  isn't a proper parameter of  $F$ , and if  $P$  is a proper parameter of  $F$ , then  $P$  isn't a proper dimension of  $F$ . For parameters, the many-to-one function goes from the space of parameters to the space of  $F$ -values. For dimensions, the many-to-one function goes from the space of  $F$ -values to the space of  $D$ -values. In this sense, dimensions stand in an inverse relationship to parameters.

Although proper dimensions and proper parameters are mutually exclusive, some properties may be both multidimensional and multiparametric. Consider health: it's plausible both that there are multiple ways of varying *wrt* health and multiple parameters that determine something's health value. Perhaps one reason that multidimensionality and multiparametricity are sometimes conflated is because many properties fall under both categories. Other common examples, such as beauty, creativity, and athleticism, seem to exhibit similar patterns.

Obviously, it's beyond the scope of this paper to analyze parameters in as much detail as dimensions. Instead, I'll focus on how dimensions and parameters relate to each other. More specifically, I'll explain how thinking about the structural relationships between dimensions and parameters can sharpen our understanding of a number of philosophical topics, including aggregation and comparability.<sup>43</sup>

### Aggregation

Some philosophers have assumed that if  $F$  is multidimensional, then the dimensions of  $F$  must **aggregate** to yield overall  $F$ -scores.<sup>44</sup> However, many multidimensional properties aren't aggregative. For example, personality is multidimensional, but the dimensions of personality (such as openness and extraversion) don't aggregate to yield "overall personality scores." Even if you have a higher value than me along every dimension of personality, that doesn't mean you have a "higher personality score" than me!

Instead, aggregation is a mark of multiparametricity. If  $F$ -values aggregate to overall  $F$ -scores, then there must be multiple parameters that are combined to determine overall  $F$ -scores. To make sense of overall  $F$ -scores, however, we need a function

<sup>43</sup> I suspect the distinction between dimensions and parameters will have further applications, such as to the literature on multidimensional adjectives. For limits of space, I won't explore these here.

<sup>44</sup> For a classic example, see Broome [1991]. For a recent example, see Hedden & Nebel [2024].

from the space of  $F$ -values to a one-dimensional scale. To denote the property associated with that scale—which we might call overall  $F$ -ness—I’ll use the convention  $F^\circ$  (pronounced ‘ $F$ -circle’). Alongside  $F^\circ$ , it’s useful to define a few other concepts:<sup>45</sup>

- the  $F$ -scale            a one-dimensional space of  $F$ -scores
- an  $F$ -score            an overall  $F$ -ness value, as a result of aggregating values along the dimensions of  $F$
- $F^\circ$                     the property whose values are  $F$ -scores

The relationships between these concepts mirror the relationships between  $F$ -space,  $F$ -values, and  $F$ . These relationships are illustrated in the following diagram:

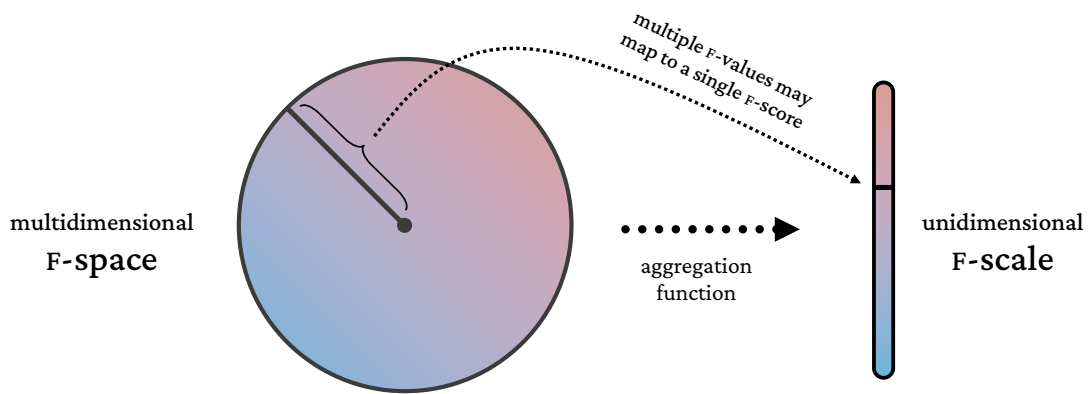


FIGURE 7:  $F$  vs.  $F^\circ$ .

Now we are in position to identify a deeper relationship between dimensions and parameters. Consider a multidimensional property  $F$  whose values aggregate to scores along property  $F^\circ$  (the  $F$ -scale). Let  $D$  be a dimension of  $F$ . What is the relationship that  $D$  stands in to  $F^\circ$ ?

The answer:  $D$  is a *parameter* of  $F^\circ$ ! To illustrate with another example, consider athleticism.<sup>46</sup> Suppose that speed, strength, and skill are each dimensions of athleticism, and that these aggregate to yield an overall athleticism score. Then there’s athleticism $^\circ$ , the unidimensional property whose values are athleticism scores. And what relation do

<sup>45</sup> Notice these definitions leave open whether overall  $F$ -scores must be interpreted as degrees of  $F$ . It’s obvious that degrees of overall  $F$ -ness are a central concern when thinking about aggregation. But it seems to me possible to think about aggregation in a more abstract way. Given this, I think it’s better to leave open what kind of structure is built into the notion of an overall  $F$ -score.

<sup>46</sup> This example features in D’Ambrosio & Hedden [2024].

speed, strength, and skill stand in to athleticism<sup>o</sup>? The answer is they are each parameters of athleticism<sup>o</sup>.

Once we distinguish  $F$  from  $F^o$ , we are in position to see that dimensions and parameters are related in systematic ways. An elegant picture emerges when we identify some of the principles connecting dimensions, parameters, properties, and scales. Suppose  $F$  aggregates to  $F^o$ . Then:

**Dimensions  $\rightleftharpoons$  Parameters**

$D$  is a dimension of  $F$   $\leftrightarrow$   $D$  is a parameter of  $F^o$ .

Therefore:

**Multidimensionality  $\rightleftharpoons$  Multiparametricity**

$F$  is multidimensional  $\leftrightarrow$   $F^o$  is multiparametric.

Now we have the conceptual resources to diagnose the Puzzling Observation from earlier. The puzzle, as we saw, arose from a conflation between dimensions and parameters. When I first drew this distinction, it may have seemed obvious: after all, the number of inputs to a function  $f$  is clearly a distinct concept from the dimensionality of  $f$ 's codomain.

But once we identify the relationship between dimensions and parameters, we can appreciate why the conflation is tempting. If  $F$  aggregates to  $F^o$ , then the dimensions of  $F$  *are* parameters of  $F^o$ . This makes it tempting to slide between dimensions and parameters. But the slide involves a subtle shift in the target kind: from  $F$  ( $F$ -space) to  $F^o$  (the  $F$ -scale). The very same property can play both the dimension role and the parameter role, but only relative to different target kinds.

The subtlety also comes from the fact that our language for talking about properties is often underspecified. When we use the term ' $F$ ', there's often systematic ambiguity between  $F$  vs.  $F^o$ , or  $F$ -space vs. the  $F$ -scale, or the space of  $F$ -values vs. the space of  $F$ -scores. This makes appeals to dimension especially precarious when theorizing about aggregation, since the dimensions of  $F$  are precisely the parameters of  $F^o$ .

**Comparability**

The distinction between dimensions and parameters can also clarify philosophical questions about **comparability**. Let's say that  $F$  is *comparable* just in case for any  $x$  and  $y$

with  $F$ -values, either  $x >_F y$  or  $x <_F y$  or  $x =_F y$ .<sup>47</sup> A number of philosophers have thought that there's a connection between multidimensionality and comparability. But what exactly is the connection?

If  $F$  is multiparametric, then there aren't any obvious consequences for comparability. After all, the mere fact that there are multiple parameters determining  $F$ -values leaves open the structure of  $F$ -space: the space of  $F$ -values could be totally ordered, or partially ordered, or even unordered. While sometimes multiple parameters aggregate to a unidimensional ordered space, other times the target space of the parameters will have a different structure.

What about multidimensionality? It may be tempting to think that multidimensionality entails incomparability. But that would be too quick. In order for comparability to even make sense, the relevant property must be degreed, and not all multidimensional properties are degreed (consider color). Furthermore, even degreed multidimensional properties can be comparable (consider velocity). Therefore, generating incomparability requires at minimum multiple degreed dimensions.

Suppose, however, that  $F$  is multidimensional, and that  $F$  has multiple degreed dimensions. Then it's plausible that the space of  $F$ -values will be at best partially ordered.<sup>48</sup> This is because there's no canonical ordering on a multidimensional space (with multiple degreed dimensions). More precisely, the intrinsic structure of  $F$ -space won't itself determine how to trade off along the dimensions of  $F$  to yield overall  $F$ -scores. Instead, we need a function from  $F$ -space to an  $F$ -scale.

If the  $F$ -scale is itself totally ordered, then there's a sense in which  $F$  will satisfy comparability (or, more accurately,  $F^\circ$  will satisfy comparability). But this means that multidimensional degreed properties will satisfy comparability only when there's the right kind of aggregation function. Furthermore, this may require a unique aggregation function: when there are multiple permissible aggregation functions from  $F$  to  $F^\circ$ , there may even still be a failure of comparability.<sup>49</sup>

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<sup>47</sup> See Hare [2010] and Dorr, Nebel, & Zuehl [2023]. Relatedly, Chang [1997, 2002] argues that some items stand in a fourth value relation ("on a par") that's distinct from better, worse, and equal. The dimensions/parameters distinction bears on this debate: the on a par relation is more naturally associated with multidimensionality than with multiparametricity.

<sup>48</sup> Multidimensionality, by itself, doesn't entail incomparability. In order for comparability to even make sense, the relevant property must be degreed, and not all multidimensional properties are degreed (consider color). Furthermore, even degreed multidimensional properties can be comparable (consider velocity). Therefore, generating incomparability requires at minimum multiple degreed dimensions.

<sup>49</sup> See D'Ambrosio & Hedden [2024].

### Multidimensionality vs. Multiparametricity

Readers who have previously used the term ‘dimension’ to mean parameter might wonder whether ‘dimension’ is polysemous, with one sense expressing parameters.<sup>50</sup> I think there are reasons to resist that conclusion.<sup>51</sup> But what matters more than the verbal question is the structural distinction between dimensions and parameters, and coordinating terminology so as to avoid confusion.

On my analysis, some of the recent literature that has used the label ‘multidimensionality’ is best understood as work on multiparametricity. This doesn’t impugn the work itself, but it does indicate a need for greater clarity in characterizing the target concept. One reason is that multiparametricity is a narrower phenomenon than the label ‘multidimensionality’ may suggest: even paradigmatic examples of multidimensional properties—such as color, spacetime location, and personality—needn’t be multiparametric. Furthermore, since the parameters of  $F^\circ$  just are the dimensions of  $F$ , even a theory of multiparametricity must appreciate the distinction between  $F$ -space and the  $F$ -scale. Therefore, even those interested principally in parameters have reason to care about my analysis of dimensions.<sup>52</sup>

If we use ‘dimension’ to express both dimensions and parameters, then we risk theoretical confusion. This is because dimensions and parameters turn out to be

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<sup>50</sup> There are interesting questions about how to think about the meanings of theoretical terms when there is a distinctive pattern of use associated with a small body of researchers (see Lee & Mankowitz [2026] for discussion in the case of ‘conscious’). I’m inclined to think that the fact that some academic subcommunities use ‘dimension’ to express parameters is enough to establish such a sense. Nevertheless, these may be technical, rather than ordinary, senses of the term.

<sup>51</sup> While it’s beyond the scope of this paper to adjudicate this semantic question, I’ll state a few brief points on this. First, as far as I can tell, the most common uses of ‘dimension’ express the target notion of this paper: a way of varying with respect to a kind. Second, as far as I can tell, the only contexts where ‘dimension’ is used to express parameter are the academic literatures I’ve mentioned (such as certain ethics and semantics literatures). Third, sentences such as ‘ $F$  is scalar / has the structure of  $\mathbb{R}$  / varies in only one respect and  $F$  is also multidimensional’ strike many as infelicitous. Fourth, in mathematics, ‘dimension’ is used to express measures of dimensionality—roughly, the number of ways of varying / degrees of freedom / directions of movement—rather than parametricity. Fifth, many paradigmatic examples of multidimensional properties—such as color and personality—aren’t multiparametric.

<sup>52</sup> For example, Hedden & Nebel [2024] define a concept  $F$  as multidimensional “just in case whether and to what extent something is  $F$  depends on how it stands along multiple underlying dimensions.” Yet this sentence seems to me to make three false claims! Even if  $F$  is multidimensional, it needn’t follow that (1)  $F$  involves aggregation, (2)  $F$  comes in degrees, or (3) whether something is  $F$  depends (non-trivially) on how it stands along the dimensions of  $F$ . Color is a counterexample to all three claims: color isn’t aggregative, color isn’t degreed, and whether something is colored doesn’t require it to have any particular combination of hue, saturation, or brightness.

systematically related—the dimensions of  $F$  are parameters of  $F^\circ$ —yet they stand in an inverse relationship. To use ‘dimension’ for both dimensions and parameters is analogous to using a single term for both truths and facts, or reasons and causes, or determinables and determinates.

The question of how to apply the label ‘dimension’ is verbal, but the distinction between dimensions and parameters is substantive. This distinction in kind ought to be mirrored by a distinction in terminology. I’ve offered a way of moving forward: distinguish ‘dimension’ from ‘parameter’.

## Conclusion

I’ve explored a number of questions about the dimension relation. At the heart of this paper is the quotient analysis: for  $D$  to be a dimension of  $F$  is for the space of  $D$ -values to be a quotient of the space of  $F$ -values.

The basic idea of a dimension—a way of varying with respect to a kind—is intuitive. Yet the concept, in my view, turns out to express a deep mathematical relation: namely, that of a quotient. This may initially feel surprising, since the mathematics of quotients can become complex and intricate. But while the local details vary from case to case, the common pattern—an operation involving abstraction and structure-preservation—turns out to be one of the most universal mathematical concepts.

The quotient analysis reveals a number of structural connections that are worth exploring. The duality between the dimension relation and the parthood relation suggests that these two relations are—in a sense that can be made formally precise—complementary ways of extracting something simple from something complex. Dimensions stand to quotients as parts stand to subobjects. The distinction between dimensions and parameters captures inverse relations between multidimensional properties and unidimensional scales. The dimensions of  $F$  are the parameters of  $F^\circ$ .

The ideas in this paper are merely the start of a larger line of inquiry. I’ve mostly set aside questions about dimensionality and dimensionalization, and I’ve only begun exploring the relationship between dimensions, parts, and parameters. But I hope I’ve illustrated how the subject-matter of dimensions is rich, fertile, and exciting. A theory of dimensions, in my view, promises to illuminate some long-standing philosophical questions—and to reveal entirely new ones.

## APPENDIX: Quotients and Partitions

In the paper, I defined quotients via structure-preserving maps. But I noted that quotients can also be defined via structure-inheriting partitions. This APPENDIX explains the latter way of thinking about quotients.

A quotient—on this approach—starts with a partition. The partition, when thinking about the dimension relation, is over the space of  $F$ -values. But a quotient not only divides that space into equivalence classes, but also specifies how the partitioned space inherits structure from its source space.

For most kinds of mathematical structures, only certain kinds of partitions count as quotients. The question is whether the partition “respects the structure of the space.” In such cases, the equivalence relation that generates the partition is called a **congruence**. When a partition is generated by a congruence, the partition inherits structure from the source space. But what counts as a congruence depends on the kind of structure under consideration.

It’s beyond the scope of this paper to explain congruences across all mathematical domains. Just as with the notion of a structure-preserving map, the exact definition varies across mathematical domains.<sup>53</sup> Because of this—and for the reasons expressed in §4—I’ll focus merely on explaining how structure-inheritance works for **relational spaces**, meaning sets of elements endowed with some relations over those elements.

For relational spaces, structure-inheritance is defined by the “universal lift:” a cell stands in a relation just in case each element in a cell stands in that relation. To denote the cell of the partition that contains value  $F_i$ , I’ll use  $[F_i]$ . Then, for any relation  $R$ , we can define  $R([F_i], \dots, [F_j])$  as holding iff  $R(F_i, \dots, F_j)$  for every combination of values across  $[F_i], \dots, [F_j]$ .<sup>54</sup> In other words, a tuple of cells stands in a relation just in case every corresponding tuple of their elements stands in that relation.

To apply this to an example, consider again hue and color. Suppose we partition color space by hue, so that  $F_1 \sim F_2$  iff  $F_1$  and  $F_2$  determine the same hue value. The resulting partition abstracts away from all differences in saturation and brightness, leaving intact only differences in hue. Each equivalence class in the partition will preserve relations between hue values while forgetting relations corresponding to brightness and

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<sup>53</sup> Here are a few prominent examples. For topological spaces, the partitioned space is equipped with the quotient topology, the finest topology that permits a continuous mapping from  $F$ -space to  $F/\sim$ . For partially ordered sets, structure-inheritance is defined by the “existential lift,” where if  $F_1 \geq F_2$ , then  $[F_1] \geq [F_2]$ . For metric spaces, structure-inheritance is defined by equipping the partitioned space with the least-distance metric, where the distance between two cells is defined as the infimum of the distances between their elements.

<sup>54</sup> More formally,  $R([F_1], \dots, [F_n]) \leftrightarrow \forall F_1 \in [F_1] \dots \forall F_n \in [F_n] \ R(F_1, \dots, F_n)$ .

saturation values. More precisely, equivalence classes of the partition will stand in the angular distances that characterize the hue circle.

There's a deep connection between quotient maps, quotient spaces, equivalence relations, and structured partitions. In brief,  $D$ -space is a quotient space of  $F$ -space just in case there's some equivalence relation  $\sim$  such that the structured partition  $F/\sim$  is **isomorphic** to  $D$ . In other words, a quotient space of  $F$  is structurally equivalent to a structure-inheriting partition of  $F$ . This idea can be expressed symbolically as follows ( $\cong$  means 'is isomorphic to'):<sup>55</sup>

$$D \text{ is a } \mathbf{quotient} \text{ of } F \quad \leftrightarrow \quad \exists (q : F \rightarrow D) : (F/\sim_q \cong D),$$

$$\text{where } F_1 \sim_q F_2 \leftrightarrow q(F_1) = q(F_2).$$

This elegant connection between quotient spaces and structured partitions justifies the two ways of defining quotients. If we care only about differences up to isomorphism, then quotient spaces and partitioned spaces are indeed distinct ways of looking at the same mathematical object.

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<sup>55</sup> This is a generalization of the First Isomorphism Theorem, a theorem in abstract algebra. See Burris & Sankappanavar [1981] on the theorem, and Awodey [2010] on the categorical generalization of quotients.

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## References

- Awodey, Steve. *Category Theory*, 2nd ed., Oxford Logic Guides 52, Oxford University Press, 2010. ISBN 978-0-19-923718-0.
- Burris, Stanley & Sankappanavar, H. P. *A Course in Universal Algebra*. Graduate Texts in Mathematics 78, Springer-Verlag, 1981.
- Broome, John. *Weighing Goods: Equality, Uncertainty and Time*. Wiley-Blackwell, 1991.
- Chang, Ruth (ed.). *Incommensurability, Incomparability, and Practical Reason*. Cambridge, MA: Harvard University Press, 1997.
- Chang, Ruth (2002). The possibility of parity. *Ethics* 112 (4):659-688.
- Calosi, Claudio & Wilson, Jessica (2021). Quantum indeterminacy and the double-slit experiment. *Philosophical Studies* 178 (10):3291-3317.
- D'Ambrosio, Justin & Hedden, Brian (2024). Multidimensional Adjectives. *Australasian Journal of Philosophy* 102 (2):253-277.
- D'Ambrosio, Justin & Stoljar, Daniel (ms). The Indeterminacy of Consciousness.
- Dorr, Cian, Nebel, Jacob M. & Zuehl, Jake (2023). The Case for Comparability. *Noûs* 57 (2):414-453.
- Ferreiros, J.; Gray, J. J. (2006-04-27). *The Architecture of Modern Mathematics: Essays in History and Philosophy*. OUP Oxford. ISBN 978-0-19-151379-4.
- Funkhouser, Eric (2006). The determinable-determinate relation. *Noûs* 40 (3):548–569.
- Green, E. J. (2020). The Perception-Cognition Border: A Case for Architectural Division. *Philosophical Review* 129 (3):323-393.
- Hare, Caspar (2010). Take the sugar. *Analysis* 70 (2): 237–247.
- Hedden, Brian & Muñoz, Daniel (2024). Dimensions of Value. *Noûs* 58 (2):291-305.
- Hedden, Brian & Nebel, Jacob M. (2024). Multidimensional Concepts and Disparate Scale Types. *Philosophical Review* 133 (3):265-308.
- Jacobs, Caspar (forthcoming). In Defence of Dimensions. *British Journal for the Philosophy of Science*.

- 
- Jalloh, Mahmoud (2023). Dimensional Analysis: Essays on the Metaphysics and Epistemology of Quantities. Dissertation, University of Southern California
- Jacobson, Hilla (forthcoming). On the Very Idea of Valenced Perception. *Journal of Philosophy*.
- Kennedy, C., & McNally, L. (2005). Scale Structure, Degree Modification, and the Semantics of Gradable Predicates. *Language*, 81.
- Kirby, Kyle (forthcoming). The Aesthetic Dimension of Value. *Australasian Journal of Philosophy*.
- Koslicki, Kathrin. *The Structure of Objects*. Oxford University Press, 2008.
- Kulvicki, John (2020). Modeling the Meanings of Pictures: Depiction and the Philosophy of Language. Oxford, GB: Oxford University Press.
- Lee, Andrew Y. (2021). Modeling Mental Qualities. *The Philosophical Review* 130 (2):263-209.
- Mac Lane, Saunders (1998). *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Vol. 5 (2nd ed.). Springer-Verlag. ISBN 0-387-98403-8. Zbl 0906.18001
- Maley, Corey (2026). Structural representation is analog representation. *Philosophy and the Mind Sciences* 7 (1).
- Mason, Sam (2025). Dimensions of Emotional Fit. *The Philosophical Quarterly* 75 (1):125-146.
- Melamedoff, Ariel (forthcoming). Neutral Monism and the Attributes of God in Spinoza's Metaphysics. *Archiv für Geschichte der Philosophie*.
- Nagel, Thomas (1979). The Fragmentation of Value. In *Mortal Questions*. New York: Cambridge University Press, pp. 128-141.
- Richeson, D. S. (2021, September 13). The journey to define dimension. *Quanta Magazine*. Retrieved from <https://www.quantamagazine.org/a-mathematicians-guided-tour-through-high-dimensions-20210913/>
- Sagan, Hans (1994), *Space-Filling Curves*, Universitext, Springer-Verlag, doi:10.1007/978-1-4612-0871-6, ISBN 0-387-94265-3, MR 1299533.
- Stevens, S. S. (June 7, 1946). "On the Theory of Scales of Measurement" (PDF). *Science*. 103 (2684): 677–680.
- Strawson, Galen. *Stuff, Quality, Structure*.

---

Varzi, Achille, "Mereology", The Stanford Encyclopedia of Philosophy (Spring 2019 Edition), Edward N. Zalta (ed.), URL = <<https://plato.stanford.edu/archives/spr2019/entries/mereology/>>.

White, P. Quinn (2025). Beautiful, troubling art: in defense of non-summativ judgment. *Philosophical Studies* 182 (3):775-799.

Wilson, Jessica, "Determinables and Determinates", The Stanford Encyclopedia of Philosophy (Spring 2023 Edition), Edward N. Zalta & Uri Nodelman (eds.), URL = <<https://plato.stanford.edu/archives/spr2023/entries/determinate-determinables/>>.

Zell E, Lesick TL. Big five personality traits and performance: A quantitative synthesis of 50+ meta-analyses. *J Pers.* 2022 Aug;90(4):559-573. doi: 10.1111/jopy.12683. Epub 2021 Nov 1. PMID: 34687041.